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On the Control of Open Quantum Systems in the Weak Coupling Limit*

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Abstract

When dealing with the control of quantum systems in interaction with an external environment, one of the most commonly used models is the Lindblad-Kossakowski equation for the density matrix. Its derivation relies on various approximations, which are usually justified in terms of weak interaction between the system and its surroundings and special properties of the bath interacting with the system. This equation contains dissipative corrections, accounting for the interaction with the environment, whose expression strongly depends on the adopted Markov approximation. In the case of coherently controlled Lindblad-Kossakowski equation, it is usually assumed that this correction is independent of the control Hamiltonian. However, this procedure is not consistent with the rigorous derivation of the Markov approximation in the standard case, the so-called weak coupling limit, in which the dissipative contribution depends on the coherent part of the dynamics.

In this paper we discuss the rigorous derivation of the Lindblad-Kossakowski equation in the weak coupling limit regime, following the original derivation of Davies [7], in the case where the control is present, and explicitly displaying the dependence of the model on the control. We then consider the special case of a 2*-level system in a bosonic bath, and show explicitly how the model depends on the parameters in the Hamiltonian which contains the control. We also discuss the impact of this more rigorous modeling procedure in the current research on methods to find controls that decouple the system from the environment. We indicate directions and problems for further research.*

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1. INTRODUCTION

In the last decades there has been a large interest in the control of quantum systems whose interaction with the environment cannot be neglected. Such systems are named *open quantum systems* and their control presents a series of challenges and a structure which is not a simple extension of what is known for the case of *closed* systems [3]. The effect of the interaction with the environment leads to a degradation of the 'quantum' nature of the evolution which jeopardizes the correct implementation of quantum information processing. Therefore control of the dynamics to shield a system from the bath is currently investigated.

Generally, a well motivated model of open quantum dynamics is the *Markovian Quantum Master Equation* for the density matrix ρ describing the system. This is given by

$$\dot{\rho} = [-iH,\rho] + \sum_{j,k}\left(V_j\rho V_k^{\dagger} - \frac{1}{2}\left\{V_k^{\dagger}V_j,\rho\right\}\right), \qquad (1)$$

where the operators V_j are the so-called *Lindblad-Kossakowski operators*, cf., e.g., [2]. Here H is the closed system Hamiltonian H_0 augmented with the so-called *Lamb shift* term, H_{Lamb}, which depends on the interaction between the system and the environment. The derivation of the equation (1) is the work of many authors including Nakajima, Zwanzig, Prigogine, Resibois, Lindblad, Kossakowski and Davies. This work proceeded from phenomenological arguments to rigorous mathematical proofs (see e.g., [5], [6], [7], and the references in [13]). Equation (1) is valid under the assumption that the coupling with the external environment is small and that memory effects are negligible, the Born and Markov approximations. More specifically, there are two main approaches to rigorous Markov approximations, valid under different conditions: the *weak coupling limit* and the *singular coupling limit*. In the first case, which is usually better motivated from a physical point of view, the Lindblad-Kossakowski operators depend on the Hamiltonian H_0. A survey on the master equation is presented in [8].

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When dealing with coherent control of open quantum systems, the Hamiltonian H_0 depends on the control and typically authors in the control theory literature modify equation (1) by simply replacing H_0 with a time varying Hamiltonian, $H_c := H_c(u)$, depending on the control (see e.g., papers in the special issue of the *IEEE Transactions on Automatic Control* dedicated to quantum control [16], and the references therein). While this procedure is justified in the *singular coupling limit*, it is not in the *weak coupling limit*, since in the latter case the V_j's operators in (1) depend on the Hamiltonian itself, which in turn depends on the control (which, in principle, has yet to be designed). By using an example, in this paper we will show that this dependence can be very significant. While this fact has been already discussed to some extent in the physics literature [14], our goal is to point it out to the control community, and to suggest directions for future research.

The paper is organized as follows. In section 2 we describe the quantum master equation and define the above mentioned approximations. Our treatment follows mainly the work of Davies in [7] (cf. also [5], [6], [7], and [4]) and it is presented in a way that highlights the dependence of the Lindblad-Kossakowski operators on the Hamiltonian and therefore the controls. We stress the role of the *Generalized Master Equation* which is an exact integral equation, of the Volterra type, which describes the evolution before any approximation is applied. The two cases of singular coupling limit and weak coupling limit are discussed, pointing out that in the second case the Lindblad-Kossakowski operators depend on the control. In section 3 we consider the model of a 2-level system in an external environment which consists of a continuous set of harmonic oscillators (*bosonic bath*). We derive the generator of the dynamics, and explicitly show the aforementioned dependence. In section 4 we discuss how this dependence may affect current research in finding controls to decouple the system dynamics from the environment (see e.g., [10], and references therein).

We would like to emphasize at the outset that, although the arguments presented in this paper limit the applicability of some of the results published in the literature, they also open new scenarios for the development of the theory of control of open quantum systems. The dependence of the model of de-coherence on the control offers a new way to shield a quantum systems from the detrimental effects of the interaction with the environment as the control enters not only the coherent part of the model, but also the additional terms modeling the interaction with the environment.

2. THE QUANTUM MASTER EQUATION

2.1. Preliminaries

We consider a system S and a bath B in a composite state described by a density operator ρ_T on the Hilbert space $\mathscr{H}_S \otimes \mathscr{H}_B$. Our objective is to obtain an equation for the evolution of the state of the system S only, which is $\rho_S := Tr_B(\rho_T)$. We assume that the states of the system S and bath B are initially uncorrelated so that

$$\rho_T(0) = \rho_S(0) \otimes \rho_B, \tag{2}$$

for an equilibrium state of the bath ρ_B, satisfying ($[H_B, \rho_B] = 0$). The dynamics of the total system $S+B$ is determined by an Hamiltonian operator $H_{TOT}(t)$, given by the sum of the term $\hat{H}_S(t) \otimes \mathbf{1}$, which describes the dynamics of the system alone, the term $\mathbf{1} \otimes H_B$ which describes the dynamics of the bath alone, and finally the term $\varepsilon \hat{H}_{SB}$, which describes the interaction between system and bath. The parameter ε is used to describe the strength of the interaction, which, in the limit considered below, is assumed to be small. We have therefore

$$H_{TOT}(t) := \hat{H}_S(t) \otimes \mathbf{1} + \mathbf{1} \otimes H_B + \varepsilon \hat{H}_{SB}. \tag{3}$$

Without loss of generality, we write the interaction Hamiltonian as $\hat{H}_{SB} := \sum_j V_{Sj} \otimes V_{Bj}$. Adding and subtracting $\sum_j Tr(V_{Bj}\rho_B) V_{S_j} \otimes \mathbf{1} = Tr_B(\hat{H}_{SB} \mathbf{1} \otimes \rho_B)$ in (3), we define

$$H_S^{\varepsilon}(t) := \hat{H}_S(t) + \varepsilon \sum_j Tr(V_{Bj}\rho_B) V_{Sj}, \tag{4}$$

$$H_{SB} := \hat{H}_{SB} - \varepsilon \sum_j Tr(V_{Bj}\rho_B) V_{S_j} \otimes \mathbf{1}, \tag{5}$$

so that we can rewrite the Hamiltonian $H_{TOT}(t)$ in (3) as $H_{TOT}(t) = H_S^{\varepsilon}(t) \otimes \mathbf{1} + \mathbf{1} \otimes H_B + \varepsilon H_{SB}$. The term $\varepsilon Tr_B(\hat{H}_{SB} \mathbf{1} \otimes \rho_B)$ in (4) is the so-called *Lamb shift*. It represents a shift in the Hamiltonian of the system due to the interaction with the bath. Notice that the Hamiltonian $\hat{H}_S(t)$ (and therefore $H_S^{\varepsilon}(t)$) can be time-dependent, since it contains the time-varying control fields; all other Hamiltonians are assumed constant.

The dynamics of ρ_T follows the Liouville-Schrödinger equation $\dot{\rho}_T = [-iH_{TOT}(t), \rho_T] = [-iH_S^{\varepsilon}(t) \otimes \mathbf{1}, \rho_T] + [-i\mathbf{1} \otimes H_B, \rho_T] + \varepsilon[-iH_{SB}, \rho_T]$.

As customary, ad_A denotes the operator $ad_A(B) := [A, B]$, and we use the short notation $ad_S^{\varepsilon}(t)$, ad_B and ad_{SB} for $ad_{-iH_S^{\varepsilon}(t) \otimes \mathbf{1}}$, $ad_{-i\mathbf{1} \otimes H_B}$, and $ad_{-iH_{SB}}$, respectively, so that this equation can be compactly written as

$$\dot{\rho}_T = (ad_S^{\varepsilon}(t) + ad_B)(\rho_T) + \varepsilon ad_{SB}(\rho_T). \tag{6}$$

Following the technique of Nakajima [12] and Zwanzig [15], we define a projection operator P (cf. [13]), as

$$P[\rho] = Tr_B(\rho) \otimes \rho_B, \tag{7}$$

so that $P(\rho_T) = \rho_S \otimes \rho_B$, where ρ_B in the definition of P is the same as in (2). We denote by $Q := \mathbf{1} - P$. It follows from the definitions and properties of $ad_S^\varepsilon(t)$, ad_B and ad_{SB}, and ρ_B that

$$Pad_S^\varepsilon(t) = ad_S^\varepsilon(t)P, \quad Qad_S^\varepsilon(t) = ad_S^\varepsilon(t)Q, \qquad \forall t \geq 0 \tag{8}$$

$$Pad_B = ad_B P = 0, \qquad Qad_B = ad_B Q = ad_B, \tag{9}$$

$$Pad_{SB}P = 0. \tag{10}$$

In particular this last one follows from (5).

2.2. The Generalized Master Equation

The *Generalized Master Equation* (GME) is a *Volterra integral equation* representing the dynamics of ρ_S. It does not involve any approximation, and it represents the starting point to obtain a Markovian dynamics, in the form of a differential equation as in (1).

By writing $ad_{SB} = (P+Q)ad_{SB}(P+Q)$, and recalling (10), we can rewrite equation (6) as

$$\dot{\rho}_T = (ad_S^\varepsilon(t) + ad_B + \varepsilon Qad_{SB}Q)(\rho_T) + \tag{11}$$

$$\varepsilon(Qad_{SB}P + Pad_{SB}Q)(\rho_T).$$

Following [5], [6], [7], we introduce evolution operators (or propagators) associated with the linear operators appearing in (11). $\Phi_0(t,s)$ is the propagator associated with $ad_S^0(t) + ad_B$, and it would describe the evolution of the state ρ_T if there was no interaction between system and bath. $\Phi_1^\varepsilon(t,s)$ is the propagator associated with $ad_S^\varepsilon(t) + ad_B$; it is the same as $\Phi_0(t,s)$ but it takes into account the Lamb-shift term in the free dynamics of the system S. $\Phi_2^\varepsilon(t,s)$ is the propagator associated with $ad_S^\varepsilon(t) + ad_B + \varepsilon Qad_{SB}Q$, cf.(11). $\Phi_3^\varepsilon(t,s)$ is the (full) propagator associated with $ad_S^\varepsilon(t) + ad_B + \varepsilon Qad_{SB}Q + \varepsilon(Qad_{SB}P + Pad_{SB}Q)$. Notice that because of (8), (9), we have

$$P\Phi_2^\varepsilon(t,s) = \Phi_2^\varepsilon(t,s)P = P\Phi_1^\varepsilon(t,s) = \Phi_1^\varepsilon(t,s)P. \tag{12}$$

The propagator Φ_3^ε satisfies the following integral equation (cf. (11))

$$\Phi_3^\varepsilon(t,s) = \Phi_2^\varepsilon(t,s) + \tag{13}$$

$$\varepsilon \int_s^t \Phi_2^\varepsilon(t,r)(Qad_{SB}P + Pad_{SB}Q)\Phi_3^\varepsilon(r,s)dr.$$

From this, by using (12), we calculate $P\Phi_3^\varepsilon(t,s)P$ and $Q\Phi_3^\varepsilon(t,s)P$; we obtain respectively

$$P\Phi_3^\varepsilon(t,s)P = P\Phi_1^\varepsilon(t,s)P + \varepsilon \int_s^t \Phi_1^\varepsilon(t,r)Pad_{SB}Q\Phi_3^\varepsilon(r,s)Pdr, \tag{14}$$

$$Q\Phi_3^\varepsilon(t,s)P = \varepsilon \int_s^t \Phi_2^\varepsilon(t,x)Qad_{SB}P\Phi_3^\varepsilon(x,s)Pdx. \tag{15}$$

Replacing (15) into (14), and setting $s = 0$, we obtain

$$P\Phi_3^\varepsilon(t,0)P = P\Phi_1^\varepsilon(t,0)P + \tag{16}$$

$$\varepsilon^2 \int_0^t \Phi_1^\varepsilon(t,r)Pad_{SB}\left[\int_0^r \Phi_2^\varepsilon(r,x)Qad_{SB}dx\right]Hdr,$$

$H := P\Phi_3^\varepsilon(x,0)P$, where the term in square brackets is $Q\Phi_3^\varepsilon(r,0)P$. We can slightly simplify (16) by using (10) to replace $Qad_{SB}P$ with $ad_{SB}P$, and obtain

$$P\Phi_3^\varepsilon(t,0)P = P\Phi_1^\varepsilon(t,0)P + \tag{17}$$

$$\varepsilon^2 \int_0^t \Phi_1^\varepsilon(t,r)Pad_{SB}\left[\int_0^r \Phi_2^\varepsilon(r,x)ad_{SB}P\Phi_3^\varepsilon(x,0)Pdx\right]dr.$$

The operator $P\Phi_3^\varepsilon(t,0)P$ in (17) describes the evolution of the state of ρ_S of S. Since the initial state of the total system $S+B$ is given by (2), P acts on it as the identity. $\Phi_3^\varepsilon(t,0)P\rho_T(0)$ gives the state of the system $S+B$ at time t, and the application of P recovers the state of the system S: the final result is $\rho_S(t) \otimes \rho_B$. Changing the order of integration in (17) and replacing $\Phi_1^\varepsilon(t,r)$ with $\Phi_1^\varepsilon(t,x)(\Phi_1^\varepsilon(r,x))^{-1}$, we obtain

$$P\Phi_3^\varepsilon(t,0)P = P\Phi_1^\varepsilon(t,0)P + \tag{18}$$

$$\varepsilon^2 \int_0^t \Phi_1^\varepsilon(t,x)\left[\int_x^t (\Phi_1^\varepsilon(r,x))^{-1}Pad_{SB}\Phi_2^\varepsilon(r,x)ad_{SB}dr\right]Hdx,$$

with $H := P\Phi_3^\varepsilon(x,0)P$. Defining as $L = L(\varepsilon,t,x)$ the kernel

$$L(\varepsilon,t,x) := \varepsilon^2 \int_x^t (\Phi_1^\varepsilon(r,x))^{-1}Pad_{SB}\Phi_2^\varepsilon(r,x)ad_{SB}dr, \tag{19}$$

equation (18) can be written in the form

$$P\Phi_3^\varepsilon(t,0)P = P\Phi_1^\varepsilon(t,0)P + \int_0^t \Phi_1^\varepsilon(t,x)L(\varepsilon,t,x)P\Phi_3^\varepsilon(x,0)Pdx, \tag{20}$$

which, when applied to $\rho_S(0) \otimes \rho_B$, gives the *Generalized Master Equation* (GME)

$$\rho_S(t) \otimes \rho_B = \Phi_1^\varepsilon(t,0)[\rho_S(0) \otimes \rho_B] + \tag{21}$$

$$\int_0^t \Phi_1^\varepsilon(t,x)L(\varepsilon,t,x)[\rho_S(x) \otimes \rho_B]dx.$$

The GME is an exact representation of the dynamics of the state of the system. There is no approximation involved at this stage. The GME is a *Volterra integral equation* [9] and shows how the state ρ_S at time t depends on the its previous evolution. In other words, the state $\rho_S(t)$ is a weighed sum of the whole history of ρ_S, and, consequently, the evolution is non-Markovian.

2.3. The Weak Coupling Limit

The work of Davies [5], [6] and Davies and Spohn [7] aimed to identify an appropriate generator (i.e., a differential equation) whose solution is a good approximation, in a precise sense, of the real trajectory as $\varepsilon \to 0$. We base our analysis on the main result (Theorem 2) of [7], which describes the generator, and is valid for 'slow' controls. Several variations are available (cf., [1], [8], [13]). This result assumes that the Lamb shift $Tr_B(\hat{H}_{SB}\mathbf{1}\otimes\rho_B)$ in (4) vanishes, so that $\hat{H}_S(t) = H_S^{\varepsilon}(t) := H_S(t)$ and $\Phi_0(t,s) = \Phi_1^{\varepsilon}(t,s)$, independently of ε. Assuming that the integral converges, defining $R(t,r) := e^{-\hat{ad}_S(t)r}\otimes e^{-ad_B r}$ consider the operator $\tilde{L}$ defined as

$$\tilde{L}(t)[\rho_S] := Tr_B\left(\int_0^{+\infty} R(t,r)ad_{SB}R(t,r)^{-1}ad_{SB}[\rho_S\otimes\rho_B]dr\right) \tag{22}$$

for every ρ_S. We have denoted by $\hat{ad}_S(t)$ the operator $\rho_S \to [-iH_S(t),\rho_S]$, which can be written as

$$\hat{ad}_S(t) := \sum_{n=1}^{n_S^2-1} -i\lambda_j(t)\Pi_j(t), \tag{23}$$

where n_S is the dimension of the system S (which we assume finite), and $\Pi_j(t)$ are the projections onto the one dimensional eigenspaces of $\hat{ad}_S(t)$. If we assume that the control $u = u(t)$ (and therefore $H_S(t)$) is an analytic function of t, it can be shown that both Π and λ_j can be taken as analytic functions. We assume this is the case in the following. We also define

$$K = K(t) = -\sum_{j=1}^{n_S^2-1} \Pi_j(t)\Pi_j'(t) \tag{24}$$

Theorem 1 *Consider the solution ρ^{ε} of the linear differential equation*

$$\frac{d\rho^{\varepsilon}}{dt} = \left(\hat{ad}_S(t) + K(t) + \varepsilon^2 L^{\natural}(t)\right)[\rho^{\varepsilon}(t)], \tag{25}$$

where

$$L^{\natural}(t) := \sum_{j=1}^{n_S^2-1} \Pi_j(t)\tilde{L}(t)\Pi_j(t) \tag{26}$$

under the above assumptions. Then $\rho^{\varepsilon}(t)$ tends to the solution $\rho_S(t)$ in (21) as $\varepsilon \to 0$ in the following sense: for a fixed τ_0

$$\lim_{\varepsilon\to 0}\sup_{0\le t\le \varepsilon^{-2}\tau_0} \|\rho_S(t) - \rho^{\varepsilon}(t)\| = 0. \tag{27}$$

For future reference, we find it convenient to rewrite the term $\tilde{L}(t)[\rho_S]$ as

$$\tilde{L}(t)[\rho_S] = \tag{28}$$

$$-\sum_{j,k}\int_0^{+\infty}\Big[Tr\Big(\rho_B V_{Bj}(r)V_{Bk}\Big)\Big(V_{Sj}(t,r)V_{Sk}\rho_S - V_{Sk}\rho_S V_{Sj}(t,r)\Big)$$

$$+Tr\Big(\rho_B V_{Bk}V_{Bj}(r)\Big)\Big(\rho_S V_{Sk}V_{Sj}(t,r) - V_{Sj}(t,r)\rho_S V_{Sk}\Big)\Big]dr$$

where we have used the definition $ad_{SB}[\cdot] = [-iH_{SB},\cdot]$ and $H_{SB} = \sum_j V_{Sj}\otimes V_{Bj}$, and we have used the notation $V_{Sj}(t,r) := e^{\hat{ad}_S(t)r}[V_{Sj}]$. From the expression (28) and (26), it is possible to show that, in the special case where H_S is time invariant, the generator in (25) is of the form (1) (where the Hamiltonian H is augmented with an extra term of the form $\varepsilon^2 H_1\otimes\mathbf{1}$). For sake of brevity we omit the details. However, a special case will be shown in the next section (where we actually will allow the Hamiltonian H_S to be time varying).

2.4. The Singular Coupling Limit

We go back now to the generalized master equation (21) and the kernel operator $L = L(\varepsilon,t,x)$ in (19). When developed using the definition of Φ_1^{ε} and Φ_2^{ε} and neglecting terms in ε of order 3 or higher, this expression contains functions which measure the correlation of observables on the bath at two different times. Such functions are called *autocorrelation functions* (cf., e.g., [2]). More precisely, the autocorrelation functions arise from the trace over the bath of the double commutator appearing in equation (18), obtained since the operator ad_{SB} is applied twice. The presence of the time propagator lead to the evaluation of the average of interaction operators taken at different times. A Markovian approximation of the dynamics can be obtained under the assumption that the bath is fast. From the mathematical point of view, this means that the bath autocorrelation functions are delta-functions in time, that is, the bath has no memory. This approach is called *Singular Coupling Limit* and this leads to a rigorous Markovian approximation of the dynamics, whose details are out of the scope of this work. The assumption of singular autocorrelation functions automatically suppresses any non-Markovian contribution in the dynamics of the relevant system, and the dissipative part of the generator of the dynamics is independent of its Hamiltonian part. Therefore, any work concerning the control of open quantum systems in terms of Lindblad-Kossakowski operators which are independent on the Hamiltonian part is ultimately addressing only those systems where the singular coupling procedure is well justified. While there are scenarios where this approach is valid, the corresponding results have a limited validity, which does not cover the entirety of Markovian dynamics. On the other side, the weak coupling procedure is more satisfactory from this perspective, since it does not require drastic assumptions on the autocorrelations of the bath, but only a weak coupling between system and environment and therefore it is more widely applied.

3. A CASE STUDY

In this section we derive the generator of the Markovian dynamics in the weak coupling limit as given in (25), for a qubit coupled with a bosonic bath through the Jaynes-Cummings Hamiltonian [1]

$$H_{SB}=\sum_k g(\omega_k)\Big(\sigma_-\otimes b^\dagger(\omega_k)+\sigma_+\otimes b(\omega_k)\Big), \quad (29)$$

where ω_k is the angular frequency of the k-th bosonic mode (harmonic oscillator), and $\varepsilon g(\omega_k)$ its coupling to the qubit. As usual, $b^\dagger(\omega_k)$ and $b(\omega_k)$ are creation and annihilation operators for the k-th mode, satisfying

$$[b(\omega_k),b(\omega_l)]=[b^\dagger(\omega_k),b^\dagger(\omega_l)]=0;\,[b(\omega_k),b^\dagger(\omega_l)]=\delta_{kl}, \quad (30)$$

and σ_+, σ_- the qubit rising and lowering operators, defined by

$$\sigma_\pm=\frac{1}{\sqrt{2}}(\sigma_x\pm i\sigma_y). \quad (31)$$

The free Hamiltonians are given by

$$H_S(t)=\frac{1}{2}\omega_0(t)\hat{\sigma}_z(t),\,H_B=\sum_k\omega_k\Big(b^\dagger(\omega_k)b(\omega_k)+\frac{1}{2}\Big), \quad (32)$$

where $\hat{\sigma}_z(t):=h_x(t)\sigma_x+h_y(t)\sigma_y+h_z(t)\sigma_z$ with $h_x^2(t)+h_y^2(t)+h_z^2(t)=1$, and therefore we write $h_x(t):=\sin\theta(t)\sin\phi(t)$, $h_y(t):=\sin\theta(t)\cos\phi(t)$, $h_z(t):=\cos\theta(t)$. Our goal here is to investigate how the the dissipative part in (25) depends on the Hamiltonian $H_S(t)$ and therefore the control in a coherent control scheme. We will see that the dependence is very significant.

We define $\hat{\sigma}_x(t)$, $\hat{\sigma}_y(t)$ such that the standard $su(2)$ algebra is satisfied at any time: $[\hat{\sigma}_k(t),\hat{\sigma}_l(t)]=2i\hat{\sigma}_m(t)$, where (k,l,m) is a cyclic permutation of (x,y,z). The standard Pauli matrices are related to these redefined (control dependent) Pauli matrices through an $SO(3)$ transformation $\mathscr{U}(t)$ such that

$$\sigma_k=\sum_l\mathscr{U}_{kl}(t)\hat{\sigma}_l(t),\qquad \hat{\sigma}_k(t)=\sum_l\mathscr{U}_{lk}(t)\sigma_l, \quad (33)$$

[1] a similar analysis can be performed with an arbitrary interaction term, and the Jaynes - Cummings Hamiltonian is a prototypical example.

where $k,l\in\{x,y,z\}$. We can write $\mathscr{U}(t)=$

$$\begin{pmatrix}-\sin\phi(t)\cos\theta(t) & -\cos\phi(t)\cos\theta(t) & \sin\theta(t)\\ \cos\phi(t) & -\sin\phi(t) & 0\\ \sin\phi(t)\sin\theta(t) & \cos\phi(t)\sin\theta(t) & \cos\theta(t)\end{pmatrix}. \quad (34)$$

Moreover, defining

$$\hat{\sigma}_\pm=\frac{1}{\sqrt{2}}(\hat{\sigma}_x\pm i\hat{\sigma}_y) \quad (35)$$

in accordance with (31), we can also write

$$\sigma_k=\sum_l\mathscr{V}_{kl}(t)\hat{\sigma}_l(t),\qquad \hat{\sigma}_k(t)=\sum_l\mathscr{V}^*_{lk}(t)\sigma_l, \quad (36)$$

where $k,l\in\{+,-,z\}$, for coefficients $\mathscr{V}_{kl}$ which can be obtained from (31), (33), (34), (35). Now, following Davies' prescription summarized in the previous section, we can evaluate the different contributions appearing in the dissipative part of the generator of the dynamics in the weak coupling limit in (25). We assume that the equilibrium state of the bath is given by the vacuum state $\rho_B=|0\rangle\langle 0|$, that is, no oscillator is initially excited. Since in our case V_{Bj} is $b(\omega_j)$ or $b^\dagger(\omega_j)$, it turns out that $Tr(\rho_B V_{Bj})=0$ for all j, and there is no Lamb-shift. Define $b(\omega_j,t):=e^{-ad_{B}t}(b(\omega_j))=e^{iH_Bt}b(\omega_j)e^{-iH_Bt}$ (cf. (22)). From (30) and (32), it follows that

$$[H_B,b(\omega_j)]=-\omega_j b(\omega_j),\qquad [H_B,b^\dagger(\omega_j)]=\omega_j b^\dagger(\omega_j). \quad (37)$$

This gives

$$b(\omega_j,t)=e^{-i\omega_jt}b(\omega_j),\qquad b^\dagger(\omega_j,t)=e^{i\omega_jt}b^\dagger(\omega_j), \quad (38)$$

and the equations

$$Tr\Big(\rho_B b(\omega_k)b(\omega_j,t)\Big)=Tr\Big(\rho_B b(\omega_k,t)b(\omega_j)\Big)=0, \quad (39)$$

$$Tr\Big(\rho_B b^\dagger(\omega_k)b^\dagger(\omega_j,t)\Big)=Tr\Big(\rho_B b^\dagger(\omega_k,t)b^\dagger(\omega_j)\Big)=0,$$

$$Tr\Big(\rho_B b^\dagger(\omega_k)b(\omega_j,t)\Big)=Tr\Big(\rho_B b^\dagger(\omega_k,t)b(\omega_j)\Big)=0,$$

$$Tr\Big(\rho_B b(\omega_k,t)b^\dagger(\omega_j)\Big)=e^{-i\omega_jt}\delta_{jk},$$

$$Tr\Big(\rho_B b(\omega_k)b^\dagger(\omega_j,t)\Big)=e^{i\omega_jt}\delta_{jk}.$$

Using these in (28), we calculate

$$\tilde{L}(t)[\rho_S]= \quad (40)$$

$$-\sum_j g^2(\omega_j)\int_0^{+\infty}\Big[\Big(\sigma_+(t,r)\sigma_-\rho_S-\sigma_-\rho_S\sigma_+(t,r)\Big)e^{-i\omega_jr}+\Big(\rho_S\sigma_+\sigma_-(t,r)-\sigma_-(t,r)\rho_S\sigma_+\Big)e^{+i\omega_jr}\Big]dr,$$

where

$$\sigma_+(t,r) \equiv e^{iH_S(t)r}\sigma_+ e^{-iH_S(t)r} \quad (41)$$

$$= \mathscr{V}_{++}(t)e^{i\omega_0(t)r}\hat{\sigma}_+(t) + \mathscr{V}_{+-}(t)e^{-i\omega_0(t)r}\hat{\sigma}_-(t) + \mathscr{V}_{+z}(t)\hat{\sigma}_z(t)$$

and

$$\sigma_-(t,r) \equiv e^{iH_S(t)r}\sigma_- e^{-iH_S(t)r} \quad (42)$$

$$= \mathscr{V}_{-+}(t)e^{i\omega_0(t)r}\hat{\sigma}_+(t) + \mathscr{V}_{--}(t)e^{-i\omega_0(t)r}\hat{\sigma}_-(t) + \mathscr{V}_{-z}(t)\hat{\sigma}_z(t).$$

Of course, $\sigma_-(t,r) = \sigma_+^\dagger(t,r)$, since $\mathscr{V}_{--}^*(t) = \mathscr{V}_{++}(t)$, $\mathscr{V}_{-+}^*(t) = \mathscr{V}_{+-}(t)$, and $\mathscr{V}_{+z}(t) = \mathscr{V}_{-z}(t)$ are real numbers. In (41) and (42) we have found it convenient to express σ_+ and σ_- in the (time-dependent) basis of the eigenstates of the free time-evolution generator. To proceed with the computation it is customary to consider the limit to a continuum of harmonic oscillators,

$$\sum_j \ldots \to \int_0^{+\infty} \ldots \rho(\omega)d\omega, \quad (43)$$

where $\rho(\omega)$ is the density of modes with angular frequency ω. For the present purposes we don't need to specify the form of this density, but we simply define

$$\int_0^{+\infty} g^2(\omega)e^{\pm i\omega r}\rho(\omega)d\omega \equiv c(r) \pm is(r). \quad (44)$$

Moreover, we define

$$\int_0^{+\infty} c(r)e^{\pm i\omega_0(t)r}dr \equiv \gamma_c(\omega_0,t) \pm i\gamma_s(\omega_0,t), \quad (45)$$

$$\int_0^{+\infty} c(r)dr \equiv \gamma_0,$$

$$\int_0^{+\infty} s(r)e^{\pm i\omega_0(t)r}dr \equiv \delta_c(\omega_0,t) \pm i\delta_s(\omega_0,t),$$

$$\int_0^{+\infty} s(r)dr \equiv \delta_0,$$

and assume that all these integrals converge. Therefore, by using (41), (42), and (45) we can write $\tilde{L}$ in (28) in terms of the functions γ_c, γ_s, δ_c, δ_s, γ_0, δ_0. The next step is to average the operator $\tilde{L}$ following the Davies' prescription (26) and obtain $L^\natural(t)$. In our case model, the eigenprojections $\Pi_n(t)$ are explicitly given by

$$\Pi_j(t)[\rho_S] = \frac{1}{2}Tr\Big(\rho_S\hat{\sigma}_j^\dagger(t)\Big)\hat{\sigma}_j(t), \quad (46)$$

with $j \in \{+,-,z\}$, and the respective eigenvalues are $0, \omega_0(t), -\omega_0(t)$. We notice that

$$\sum_n \Pi_n(t)L(t)\Pi_n(t) = \lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T} e^{-iH_S(t)r}L(t)e^{iH_S(t)r}dr, \quad (47)$$

and, moreover,

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T} e^{\pm i\omega_0(t)r}dr = \lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T} e^{\pm 2i\omega_0(t)r}dr = 0. \quad (48)$$

Therefore, if we expand σ_+ and σ_- as in (41) and (42), the only non-vanishing averages contain both $\hat{\sigma}_+(t)$ and $\hat{\sigma}_-(t)$, or $\hat{\sigma}_z(t)$ twice, and we obtain

$$L^\natural(t)[\rho_S] = \quad (49)$$

$$\mathscr{V}_{++}(t)\mathscr{V}_{--}(t)\Big(\gamma_c(\omega_0,t)+\delta_s(\omega_0,t)\Big)\mathscr{A}+$$

$$+\mathscr{V}_{+-}(t)\mathscr{V}_{-+}(t)\Big(\gamma_c(\omega_0,t)-\delta_s(\omega_0,t)\Big)\mathscr{B}+$$

$$\mathscr{V}_{+z}(t)\mathscr{V}_{-z}(t)\gamma_0\Big(2\hat{\sigma}_z(t)\rho_S\hat{\sigma}_z(t)-2\rho_S\Big)\Big]+$$

$$2i\Big[\mathscr{V}_{++}(t)\mathscr{V}_{--}(t)\Big(\gamma_s(\omega_0,t)-\delta_c(\omega_0,t)\Big)[\hat{\sigma}_+(t)\hat{\sigma}_-(t),\rho_S]$$

$$-\mathscr{V}_{+-}(t)\mathscr{V}_{-+}(t)\Big(\gamma_s(\omega_0,t)+\delta_c(\omega_0,t)\Big)[\hat{\sigma}_-(t)\hat{\sigma}_+(t),\rho_S]\Big],$$

with $\mathscr{A} := \Big(2\hat{\sigma}_-(t)\rho_S\hat{\sigma}_+(t) - \{\rho_S,\hat{\sigma}_+(t)\hat{\sigma}_-(t)\}\Big)$, $\mathscr{B} := \Big(2\hat{\sigma}_+(t)\rho_S\hat{\sigma}_-(t) - \{\rho_S,\hat{\sigma}_-(t)\hat{\sigma}_+(t)\}\Big)$. In this expression, the first three terms are of the Lindblad-Kossakowski form, while the last two contributions are dissipative corrections to the coherent part of the dynamics. Using the explicit expression for the matrix $\mathscr{V}$ we can also obtain $\mathscr{V}_{++}(t)\mathscr{V}_{--}(t) = \frac{1}{2}\Big(\cos^2\frac{\theta(t)}{2} + \cos\theta(t)\cos\phi(t)\Big)$, $\mathscr{V}_{+-}(t)\mathscr{V}_{-+}(t) = \frac{1}{2}\Big(\cos^2\frac{\theta(t)}{2} - \cos\theta(t)\cos\phi(t)\Big)$, $\mathscr{V}_{+z}(t)\mathscr{V}_{-z}(t) = \frac{1}{2}\sin^2\theta(t)$, and these relations can further simplify (49). For instance, the Hamiltonian part can be more compactly rewritten as

$$2i\Big(\gamma_s(\omega_0,t)\cos\theta(t)\cos\phi(t) - \delta_c(\omega_0,t)\cos^2\frac{\theta(t)}{2}\Big)[\hat{\sigma}_z(t),\rho_S]. \quad (50)$$

The next step is the evaluation of the term $K(t)$ (24) in the Markovian generator. In the present case, it is given by the sum of three contributions: $-K(t)[\rho_S] = \Pi_0(t)\Pi_0'(t)[\rho_S] + \Pi_+(t)\Pi_+'(t)[\rho_S] + \Pi_-(t)\Pi_-'(t)[\rho_S]$, which we now compute separately. From (46) we can write

$$\Pi_0'(t)[\rho_S] = \frac{1}{2}\Big(\rho_S\hat{\sigma}_z'(t)\Big)\hat{\sigma}_z(t) + \frac{1}{2}\Big(\rho_S\hat{\sigma}_z(t)\Big)\hat{\sigma}_z'(t), \quad (51)$$

and then

$$\Pi_0(t)\Pi_0'(t)[\rho_S] = \quad (52)$$

$$\frac{1}{2}\Big(Tr\Big(\rho_S\hat{\sigma}_z'(t)\Big) + \frac{1}{2}Tr\Big(\rho_S\hat{\sigma}_z(t)\Big)Tr\Big(\hat{\sigma}_z(t)\hat{\sigma}_z'(t)\Big)\Big)\hat{\sigma}_z(t)$$

$$= \frac{1}{2}Tr\Big(\rho_S\hat{\sigma}_z'(t)\Big)\hat{\sigma}_z(t),$$

since

$$Tr\Big(\hat{\sigma}_z(t)\hat{\sigma}_z'(t)\Big) = 0 \tag{53}$$

as a consequence of $\hat{\sigma}_z^2(t) = \mathbf{1}$. On the other hand, it is possible to prove that

$$\Pi_+(t)\Pi_+'(t)[\rho_S] = \tag{54}$$

$$\frac{1}{2}Tr\Big(\rho_S\hat{\sigma}_-'(t)\Big)\hat{\sigma}_+(t) + \frac{1}{4}Tr\Big(\rho_S\hat{\sigma}_-(t)\Big)Tr\Big(\hat{\sigma}_-(t)\hat{\sigma}_+'(t)\Big)\hat{\sigma}_+(t)$$

$$= \frac{1}{2}Tr\Big(\rho_S\hat{\sigma}_-'(t)\Big)\hat{\sigma}_+(t)$$

since

$$Tr\Big(\hat{\sigma}_-(t)\hat{\sigma}_+'(t)\Big) = Tr\hat{\sigma}_z'(t) = 0, \tag{55}$$

and similarly

$$\Pi_-(t)\Pi_-'(t)[\rho_S] = \frac{1}{2}Tr\Big(\rho_S\hat{\sigma}_+'(t)\Big)\hat{\sigma}_-(t). \tag{56}$$

It is possible to prove that (52), (54) and (56) are time-dependent Lindblad-Kossakowski terms.

The above expressions show how the generator in (25) depends on the coherent Hamiltonian and therefore the control. All Kossakowski-Lindblad operators are 'rotated' according to the operator $\hat{\sigma}_z(t)$ in (32), which depends on the control.

4. System-bath decoupling and the DSM problem

A *Decoherence Splitting Manifold (DSM)* is a subset of the set of density operators $\{\rho_S = \rho_S^\dagger, \rho_S \geq 0, \mathrm{Tr}(\rho_S) = 1\}$ in which some (possibly multiple) eigenvalues $\lambda_{k\in K}$ of ρ_S are preserved, along with their multiplicities $m_{k\in K}$, while the other eigenvalues $\lambda_{\bar{k}\in\bar{K}}$ are not specified. The only requirement is that their multiplicities $m_{\bar{k}\in\bar{K}}$ remain constant. Such a subset will be denoted as $\mathscr{D}_{\Lambda_K, m_{\bar{K}}}$, where $\Lambda_K =$ block diag$\left\{\lambda_k I_{m_k} : k \in K\right\}$ denotes the diagonal matrix of preserved eigenvalues, while $m_{\bar{K}}$ denotes the specifications on the multiplicities of the remaining eigenvalues. It can be shown that $\mathscr{D}_{\Lambda_K, m_{\bar{K}}}$ is a real-analytic manifold [11]. This manifold is referred to as *splitting*, because any density matrix $\rho \in \mathscr{D}_{\Lambda_K, m_{\bar{K}}}$ has its eigenvalues *splitting* between, on the one hand, those eigenvalues Λ_K that are preserved and, on the other hand, those eigenvalues $\Lambda_{\bar{K}}$ allowed to drift. Every density matrix with eigenvalue specifications belongs to one DSM. While in unitary dynamics, all the eigenvalues of the density matrix ρ_S are preserved, in dissipative open system dynamics these eigenvalues drift. However, if the dynamics is such as to keep the system on a DSM, some of the eigenvalues are preserved, so that some information on the initial density matrix is preserved. The *DSM control problem* consists of finding a coherent control to keep ρ_S on a DSM.

Assume $\rho_S(t) \in \mathscr{D}_{K, m_{\bar{K}}}$ for every t. Then we can project it to a sub-density matrix which has unitary evolution. Let

$$\rho_S(t) = V(t)\begin{pmatrix} \Lambda_K & 0 \\ 0 & \Lambda_{\bar{K}}(t) \end{pmatrix}V^\dagger(t) \tag{57}$$

where the eigenvalues in Λ_K are preserved, while the remaining eigenvalues in $\Lambda_{\bar{K}}(t)$ could be evolving, with the only restriction that there are no eigenvalue crossings. If $V_K(t)$ is the matrix of eigenvectors for the eigenvalues of Λ_K, we define the sub-density

$$\rho_{\mathrm{DPS}}(t) := \frac{1}{\mathrm{Trace}(\Lambda_K)}V_K(t)\Lambda_K V_K^\dagger(t) \tag{58}$$

$$= \frac{1}{\mathrm{Trace}(P_{\mathrm{DPS}}(t)\rho(t)P_{\mathrm{DPS}}(t))}P_{\mathrm{DPS}}(t)\rho(t)P_{\mathrm{DPS}}(t)$$

where

$$P_{\mathrm{DPS}}(t) = V_K(t)V_K^\dagger(t).$$

If we differentiate (58), we get

$$\dot{\rho}_{\mathrm{DPS}}(t) = -\imath[H_{\mathrm{EFF}}(t), \rho_{\mathrm{DPS}}(t)], \tag{59}$$

where H_{EFF} is some "effective" Hamiltonian,

$$H_{\mathrm{EFF}}(t) := -\imath V_K(t)\dot{V}_K^\dagger(t) = \imath\dot{V}_K(t)V_K^\dagger(t) \tag{60}$$

which is easily seen to be Hermitian.

Consider now the evolution of ρ_S given in (1). It is convenient to express the matrix ρ_S in terms in the *vector of coherences representation*, i.e., using the vector $\vec{x} := [x_1, \ldots, x_{n_S}]^T$,[2] where $\rho_S = \frac{1}{n_S}\mathbf{1} + \sum_{j=1}^{n_S} x_j\sigma_j$ for some orthonormal basis $\{\sigma_j\}$, $j = 1, \ldots, n_S$ in the space of $n_S \times n_S$ Hermitian matrices. With this notation, consider the quantum master equation (1) where, in the coherent part, we assume m control variables $u_1, \ldots, u_m$, and we assume for simplicity that the control appears linearly. The quantum master equation can be written in the form

$$\dot{x} = A\vec{x} + \sum_{j=1}^{m} B_j\vec{x}u_j + G\vec{x},$$

for appropriate $(n_S^2 - 1) \times (n_S^2 - 1)$ matrices A, B_j, $j = 1, \ldots, m$, and G.

The condition that ρ_S belongs to the DSM can be expressed in terms of appropriate functions of the vector $\vec{x}$ equal to zero. These functions represent the characteristic polynomial of ρ_S, $\det(\rho_S - \lambda\mathbf{1})$ and (possibly) some of its derivatives with respect to λ, equal to

[2] n_S denotes the dimension of the system S.

zero. The constancy of the multiplicities on the complementary eigenvalues is somewhat more complicated. The lack of numerical specifications on those eigenvalues require then to be eliminated using the Tarski-Seidenberg quantifier elimination, resulting in additional polynomial constraints. Taking the Jacobian matrix of these constraints J_x, the condition for $\rho_S(t)$ to belong to a DSM at all $t \geq 0$, is

$$J_x(A\vec{x}(t) + \sum_{j=1}^{m} B_j\vec{x}(t)u_j + G\vec{x}(t)) = 0. \tag{61}$$

However it is a general fact that on the DSM

$$J_xA\vec{x} = 0, \quad \texttt{and} \quad J_xB_j\vec{x} = 0, \quad \forall j = 1, \ldots, m. \tag{62}$$

This is a a simple consequence of the fact that the dynamics without the dissipative part $G\vec{x}$ is unitary and therefore preserves all eigenvalues including the ones in the set Λ_K defining the DSM $\mathscr{D}_{\Lambda_K, m_{\bar{K}}}$. Therefore the condition (61) reduces to

$$J_x(G\vec{x}(t)) = 0. \tag{63}$$

This tells us that it is not possible to simply use the control to *cancel* all the terms in (61). The control however affects $x(t)$ in (63) and one could in principle try to use it to induce a trajectory satisfying (63) and therefore preserving the desired eigenvalues.

The above considerations assume that G in (63) is constant, independent of the control u. As we have discussed in the previous sections, this assumption is only justified in the singular coupling limit. In the weak coupling limit, G in (63) depends explicitly on the control. In that case, the situation might in fact be more favorable as it might be possible to *design* $G(u)$ so that (63) is verified.

5. Concluding Remarks

The model of the quantum master equation often used in controllability analysis of open quantum systems is valid only in the restrictive singular coupling limit, where an infinitely fast dynamics for the bath is assumed. In the more common weak coupling limit, the coherent part of the dynamics and therefore the control appears in the whole dynamical model including the part modeling dissipation. One example discussed here is a system consisting of a spin $\frac{1}{2}$ particle in a bosonic bath showing how, under the assumptions of Davies' results [7], the model depends on the control. In general the design of the control has to take into account the whole modeling procedure and what kind of assumptions on the system and bath are made. However, as we have pointed out in section 4, this might become an opportunity for future research.

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