Information Transfer Fidelity in Networks of Spins

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Abstract

Networks of spins, or spintronic networks, are given an Information Transfer Fidelity (ITF) derived from an upper bound on the probability of transmission of the excitation from one spin to another. It is shown that this theoretical bound can be reached asymptotically in time under certain conditions. The process of achieving maximum transfer probability is given a dynamical model, the translation on the torus, and the time to reach the maximum probability is estimated using the simultaneous Diophantine approximation computationally implemented using a variant of the Lenstra-Lenstra-Lovász (LLL) algorithm. The ITF induces a prametric on the network. For a ring with homogeneous couplings, it is shown that this prametric satisfies the triangle inequality, opening up the road to an ITF geometry, which turns out to be completely different from the geometry of the physical arrangement of the spin in the spintronic device. It is shown that transfer fidelities and transfer times can be improved by means of simple controls taking the form of strong localized magnetic fields, opening up the possibility for intelligent design of spintronic networks and dynamic routing of information encoded in such networks. The approach is much more flexible than engineering the couplings to favor some transfers.

Index Terms

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I. INTRODUCTION

Spintronics — or spin transport electronics — is an emerging technology aimed at utilizing not only the charge of electrons, as in conventional electronics, but also their spin degrees of freedom. This has numerous potential advantages including higher data processing speeds, decreased volatility, lower electric power consumption and decreased heat dissipation, enabling increased integration densities compared with conventional semiconductor devices [1]. The relatively-long intrinsic coherence lifetimes of spin states also makes them a potentially attractive route towards quantum information processing [2].

While successful incorporation of spin degrees of freedom into existing semiconductor technology still poses numerous technical challenges in terms of injection, transport, controlled manipulation and detection of spin polarization, considerable technical progress has been made recently. However, the information-theoretic properties of spintronic networks, for example, how information encoded in excitations of a simple network of coupled spins propagates, even under ideal conditions under which quantum coherence is maintained, has not been studied extensively. This propagation of excitations determines the Information Transfer Fidelity (ITF) between various nodes of the network.

Previously [3], this concept was named Information Transfer Capacity, but we refrain here from using this terminology to avoid confusion with the Shannon channel capacity, typically applied in the context of bosonic channels using photons as information carriers [4], [5]. The latter are attractive for long-distance communication, either in free space or via fibre optics, but efficient on-chip interconnectivity in spintronic devices will require direct information transport via spin channels. In the first instance spin transport may be achieved via charge transport in the form of spin-polarized currents. However, it is desirable to realize spin transport without charge transport to avoid speed limits imposed by RC delays and the heat dissipation related to electronic currents. An ultimate aim is quantum spintronics and the encoding of information in excitations of single spins. This motivates the study of how such excitations propagate in a network of coupled spins.

In this work, we study the intrinsic information transfer fidelities in ideal spintronic networks.
subject to coherent dynamics in general, and specifically simple network configurations such as a circular arrangement of spins (or spin ring for short), which could serve as a basic building block for more complex architectures. We introduce the concept of asymptotic ITF between nodes in a network of spins\(^1\) to define upper bounds for the intrinsic ITFs achievable by a particular network described by a given fixed Hamiltonian. We further derive conditions for attainability of the bounds using dynamic flows on tori and the simultaneous Diophantine approximation computationally implemented using the Lenstra-Lenstra-Lovász (LLL) algorithm. The ultimate aim of this analysis is to understand the intrinsic limits of such networks and utilize this understanding to engineer networks with favorable bounds on the information transfer fidelities and dynamic attainability properties, so that high spin transfer fidelities can be attained in short times, enabling fast transfer and minimizing the effects of noise and decoherence. The analysis shows that without careful design of the network the intrinsic transfer fidelities are generally limited. While the bounds are often attainable asymptotically in time, the times required to achieve high fidelities tend to be very long. The intrinsic bounds on the ITFs and transfer times can be favorably changed, however, by simple dynamic Hamiltonian engineering such as applying spatially distributed static bias fields.

II. BASIC DEFINITIONS AND RESULTS

We consider networks of \(N\) spins arranged in some regular pattern with either XX or Heisenberg interaction specified by the Hamiltonian

\[
H = \sum_{i,j=1}^{N} J_{i,j} \left( \sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y + \epsilon \sigma_i^z \sigma_j^z \right).
\]

We specifically focus on networks with XX coupling (\(\epsilon = 0\)) and Heisenberg coupling (\(\epsilon = 1\)), although most of the concepts and analysis in the following are not limited to these types of coupling. \(J_{i,j}\) is the strength of the coupling between spin \(#i\) and spin \(#j\) and is usually proportional to the cubic power of the physical distance between the two spins. The factor \(\sigma_i^{x,y,z}\) is the Pauli matrix along the \(x, y,\) or \(z\) direction of spin \(#i\), i.e.,

\[
\sigma_i^{x,y,z} = I_{2 \times 2} \otimes \ldots \otimes I_{2 \times 2} \otimes \sigma_i^{x,y,z} \otimes I_{2 \times 2} \otimes \ldots \otimes I_{2 \times 2},
\]

\(^1\)We deliberately refrain from using the terminology of “spin network” since the latter could be confused with Penrose’s quantum gravity concept, which is totally different.
where the factor $\sigma_{x,y,z}^{i}$ occupies the $i$th position among the $N$ factors and $\sigma_{x,y,z}$ is either of the single spin Pauli operators

$$
\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

Clearly, the system Hilbert space on which $H$ acts is conveniently taken as $\mathbb{C}^{2^N}$. It is convenient to abstract this network of spins as a graph $G = (\mathcal{V}, \mathcal{E})$, where the vertices represent the spins and the edges indicate the presence of couplings.

In this paper we will specifically consider spin rings, i.e., spintronic networks defined by a circular arrangement of spins, described by a $J$-coupling matrix that is circulant with nearest neighbor coupling:

$$
H = \sum_{i=1}^{N-1} J_{i,i+1} \left( \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \epsilon \sigma_i^z \sigma_{i+1}^z \right) + J_{N,1} \left( \sigma_N^x \sigma_1^x + \sigma_N^y \sigma_1^y + \epsilon \sigma_N^z \sigma_1^z \right).
$$

(1)

The term $J_{N,1}$ represents the coupling energy between the two ends, spin #1 and #N, closing the ring.

A. Single Excitation Subspace

Throughout this paper we shall restrict ourselves to the single excitation subspace, i.e., it is assumed that the total number of excitations in the network is one. The Hilbert space of the network is therefore spanned by the $N$ single excitation quantum states $\{|i\rangle: i = 1, \ldots, N\}$, where $|i\rangle = |\uparrow\uparrow\ldots\uparrow\downarrow\uparrow\ldots\uparrow\rangle$ indicates that spin #i carries the excitation.

The natural coupling among the spins allows the excitation at $i$ to drift towards an excitation at $j$ with an Information Transfer Fidelity (ITF) that can be quantified by the maximum transition probability $p_{\text{max}}(i,j)$. This concept will be precisely defined in the next section, but in this introductory exposition we could think of “maximum” as the process of giving the transition from $|i\rangle$ to $|j\rangle$ the correct amount of time so that it is most likely to occur. The concepts behind these ideas are lying at the foundation of quantum mechanics as embodied in the Feynman path integral. These concepts reveal that, contrary to classical least-cost-path routing that follows a single path from a source to a destination in a classical network, quantum networks follow all possible paths from the source $i$ to the destination $j$. From Schrödinger’s equation with an
Hamiltonian structured as (1), the transmission of the excitation is possible under the natural Hamiltonian coupling, but this can result in long transmission times or low ITF. Control here is utilized to make the ITF larger and/or the transmission time shorter. The control actions can conveniently be taken as local magnetic fields, offering more flexibility than engineering the couplings.

B. Eigendecomposition of the Hamiltonian

We shall denote the eigen-decomposition of the Hamiltonian restricted to the single excitation subspace by

$$\tilde{H} = \sum \lambda_k \Pi_k$$

where $\lambda_k$ for $k = 1, \ldots, \tilde{N} \leq N$ are the distinct real eigenvalues and $\Pi_k$ are the projectors onto the corresponding eigenspaces.

For a spin ring of size $N$ with uniform XX-coupling between adjacent spins, the single excitation subspace Hamiltonian in the basis $|i\rangle = (0, 0, \ldots, 0, 1, 0, \ldots, 0)^T$ becomes the circulant matrix

$$\tilde{H}_N = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 1 \\
1 & 0 & 1 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 \\
1 & 0 & 0 & \cdots & 1 & 0 \\
\end{pmatrix} \quad (=: C_N) \quad (2)$$

For uniform Heisenberg coupling the Hamiltonian is the same except for a multiple of the identity, which simply shifts the eigenvalues by a constant and does not affect the eigenvector structure or differences between eigenvalues. The eigenvalues and eigenvectors of circulant matrices are well known and shown in Table I. The $N$ single excitation eigenvalues are conveniently parameterized by an integer $k$ running from 0 to $\tilde{N} - 1$ or 1 to $N$ with the cyclic condition that $\lambda_{k=0} = \lambda_{k=N}$.

Application to Rings. The following lemma regarding the eigenvalues will be helpful later.

Lemma 1: For a spin ring of size $N$ with uniform XX-couplings we have:

- For $N$ even, but not divisible by 4, the spectrum of $\tilde{H}_N$ has mirror symmetry relative to the origin; precisely, there are $N/2 - 1$ distinct pairs of double eigenvalues and two single
TABLE I

EIGENVALUES AND EIGENVECTORS OF HAMILTONIAN $\bar{H}$ OVER SINGLE EXCITATION SUBSPACE [6] IN THE BASIS WHERE $|i\rangle := (0...010...0)^T$. $\rho_N$ IS THE NTH ROOT OF UNITY AND $|w_k\rangle_j$ DENOTES THE jTH COMPONENT OF $|w_k\rangle$.

| XX-coupling ($\epsilon = 0$) | $\lambda_{k=0,...,N-1}$ | $|w_k\rangle_{j=0,...,N-1}$ |
|-----------------------------|--------------------------|-----------------------------|
| $2 \cos \left(\frac{2\pi k}{N}\right)$ | $\sqrt{\frac{1}{N}}\rho_N^{k(j-1)}$ |

$\lambda_k = \ldots, N - 1$, $|w_k\rangle = \rho_N^{k(j-1)}$.

If $N$ is divisible by 4 then the spectrum still has a total of $\bar{N} = (N+2)/2$ pairwise distinct eigenvalues but it has a double eigenvalue at 0 (for $k = N/4, 3N/4$).

• For $N$ odd there are $(N-1)/2$ distinct pairs of double eigenvalues and a single eigenvalue $+2$, i.e., a total of $\bar{N} := (N+1)/2$ distinct eigenvalues:

$$\left\{-2, (\lambda_k = \lambda_{N-k}) = (-\lambda_{N/2-k} = -\lambda_{N/2+k}), +2 : k = 1, \ldots, N/2 - 1 \right\}.$$ 

In either case, the eigenvalues of $C_N$ and $C_{N-1}$ are interlaced.

Proof: The bulleted items are trivial. The last claim is the Cauchy interlacing property [7].

Regardless of whether $N$ is even or odd, for a double eigenvalue $\lambda_k = \lambda_{N-k}$, denote the projection on the corresponding eigenspace as $\Pi_k := |w_k\rangle \langle w_k| + |w_{N-k}\rangle \langle w_{N-k}|$, where the eigenvectors can be chosen such that $w_{N-k} = w_k^*$. Moreover, for the single eigenvalue $\lambda_0 = +2$, define $\Pi_0 := |w_0\rangle \langle w_0|$ to be its eigenprojection. If $N$ is even, the single eigenvalue $\lambda_{N/2} = -2$ has its eigenprojection denoted as $\Pi_{N/2} := |w_{N/2}\rangle \langle w_{N/2}|$. If, in addition, $N$ is divisible by 4, denote the eigenprojection of the double eigenvalue $\lambda_{N/4} = \lambda_{3N/4} = 0$ as $\Pi_{N/4} := |w_{N/4}\rangle \langle w_{N/4}| + |w_{3N/4}\rangle \langle w_{3N/4}|$. With this notation, the Hamiltonian restricted to the single excitation subspace...
can be written as
\[
\bar{H} = \sum_{k=0}^{N-\frac{1}{2}} \lambda_k \Pi_k,
\]
where \([N-\frac{1}{2}] + 1 = [N+\frac{1}{2}] =: \tilde{N}\) is the number of pairwise distinct eigenvalues. The above can easily be extended to the Heisenberg case by a global shifting of the eigenvalues by 1.

III. Maximum Transfer Fidelity and Attainability

Let \(|i\rangle \in \mathcal{H}\) be the quantum state where the excitation is at spin \#i. The quantum mechanical probability of transition from state \(|i\rangle\) to state \(|j\rangle\) in an amount of time \(t\) is given by
\[
p_t(|i\rangle, |j\rangle) = |\langle i|e^{-i\bar{H}t}|j\rangle|^2,
\]
where the system of units is such that \(\hbar = 1\). This formula is a corollary of the Feynman path integral [8], [9]. To circumvent the difficulty posed by the time-dependence of this probability we proceed as in [3] and define the maximum transition probability or Information Transfer Fidelity (ITF):
\[
p_{\text{max}}(i, j) = \left(\sum_{k=0}^{\tilde{N}-1} \langle i|\Pi_k|j\rangle e^{-i\lambda_k t}\right)^2
\]
\[
\leq \left(\sum_{k=0}^{\tilde{N}-1} |\langle i|\Pi_k|j\rangle|\right)^2 =: p_{\text{max}}(i, j).
\]
Clearly, \(p_{\text{max}}(i, j) \leq 1\).

A. Attainability of Bounds

The Information Transfer Fidelity \(p_{\text{max}}(i, j)\) is an upper bound on \(p_t(i, j)\), which acquires its full significance if \(p_{\text{max}}\) is achievable, that is, if there exists a sequence of time samples \(\{t_{i,j}(n) : n \in \mathbb{N}\}\) such that \(\lim_{n \to \infty} p_{t_{i,j}(n)}(i, j) = p_{\text{max}}(i, j)\). Observing that the absolute value in Eq. (3) will absorb any global phase factor, the attainability condition is that there exists a \(t \in [0, \infty)\) such that
\[
e^{-i\lambda_k t} = s_k e^{i\phi}, \quad \forall k,
\]
where \(s_k(i, j) := \text{Sgn}(|\langle i|\Pi_k|j\rangle|) \in \{0, \pm 1\}\) is a sign factor and \(\phi\) is an arbitrary global phase, which can be anything but must be the same for all \(k\)'s. Eigenspaces with \(s_k = 0\) have no
overlap with the initial and/or target state and do not contribute to the sum. We shall refer to them as *dark state* subspaces. They can be ignored and we can restrict ourselves to the set \( K' \subseteq \{0,1,\ldots,\tilde{N} - 1\} \) of indices \( k \) for which \( s_k \neq 0 \). The physical interpretation of \( K' \) is the set of eigenspaces \( \Pi_k \mathcal{H} \) that have non-trivial overlap with the initial and target state. Noting that \( s_k = \pm 1 \) for \( k \in K' \), and \( \exp[-i\pi/2(s_k - 1)] = 1 \) for \( s_k = 1 \) and \( \exp[-i\pi/2(s_k - 1)] = -1 \) for \( s_k = -1 \), we can write

\[
s_k = \exp[-i\pi(2n_k + \frac{1}{2}(s_k - 1))], \quad \forall k \in K',
\]

where \( n_k \in \mathbb{Z} \) is an arbitrary integer. Inserting this into (4), taking the logarithm and dividing by \( -i \) yields

\[
\lambda_k t = 2\pi n_k + \frac{\pi}{2}(s_k - 1) - \phi, \quad \forall k \in K'.
\]

This condition is not directly useful as \( \phi \) can be arbitrary, but we obtain meaningful constraints if we subtract the equations in a pairwise manner, with \( k \neq \ell \):

\[
(\lambda_k - \lambda_\ell)t = 2\pi(n_k - n_\ell) + \frac{\pi}{2}(s_k - s_\ell), \quad \forall k, \ell \in K'.
\]

We can also write the attainability constraints more explicitly:

\[
\begin{align*}
(\lambda_k - \lambda_\ell)t &= 2\pi(n_k - n_\ell), & \text{if } s_\ell &= s_k, \\
(\lambda_k - \lambda_\ell)t &= 2\pi(n_k - n_\ell) + \pi, & \text{if } s_k &= -s_\ell = 1, \\
(\lambda_k - \lambda_\ell)t &= 2\pi(n_k - n_\ell) - \pi, & \text{if } s_k &= -s_\ell = -1.
\end{align*}
\]

These conditions are necessary and sufficient for attainability. They are physical, only involving differences of the eigenvalues, which are observable and independent of arbitrary phases. Vanishing left-hand sides in the above are not an issue, as we are only looking at the differences, which are non-zero by definition as \( \lambda_k, k \in K' \), are the distinct eigenvalues of \( \bar{H} \).

Observe that all of the equations are compatible. Indeed, adding (7) for \((k, \ell)\) and (7) for \((\ell, m)\) yields (7) for \((k, m)\). Naturally, these equations are redundant, but if we exclude the dark state subspaces and restrict (7) to

\[
(k, \ell) = (0, 1), (1, 2), \ldots, (\tilde{N} - 2, \tilde{N} - 1),
\]

we obtain a set of linearly independent equations.
Application to Rings. For our ring system with uniform XX coupling, the distinct eigenvalues are $\lambda_k = 2 \cos(2\pi k/N)$. For single eigenvalues with 1-dimensional eigenspace, which occur for $k = 0$, and $k = N/2$ if $N$ is even, we have $\langle i|\Pi_0|j \rangle = (1/N) \neq 0$ and $\langle i|\Pi_{N/2}|j \rangle = (1/N)(-1)^{i-j} \neq 0$; therefore, there are no dark states associated with the single eigenvalues. For the multiple eigenvalues, $\langle i|\Pi_k|j \rangle = 2N \cos(\pi/2x)$ with $x = 4k(i-j)/N$ for $k = 0, \ldots, [(N-4)/2]$; therefore there are dark states if and only if $x$ is an odd integer. This can happen only if $N$ is divisible by 4. The same holds of rings with uniform Heisenberg coupling as they have the same eigenspace structure and the differences between eigenvalues are the same.

B. Simultaneous Attainability and Flows on the Torus

Excluding dark state subspaces, restricting (7) to pairs of the form (8), dividing (7) by $\pi$, and setting $\omega_{k,\ell} = (\lambda_k - \lambda_\ell)/\pi$, the attainability conditions become

$$\omega_{k,\ell}t = \frac{1}{2}(s_k - s_\ell) \mod 2, \quad k + \ell = 1, 3, 5, \ldots, 2\tilde{N} - 3. \tag{9}$$

The left-hand side of the above is the solution of the flow on the torus $\dot{x} = \omega$, with $x(0) = 0$. In this dynamical set-up, the question is whether the flow starting at $x(0) = 0$ passes through the point with coordinates 0 or 1, depending on whether $s_k = s_\ell$ or $s_k \neq s_\ell$, respectively. It is well-known [10, Prop. 1.5.1] that the flow starting at an arbitrary $x(0)$ (which includes $x(0) = 0$) passes arbitrarily close to an arbitrary point on the torus if and only if the $\omega_{k,\ell}$’s are linearly independent over the rationals $\mathbb{Q}$. This property of the flow getting arbitrarily close to an arbitrary point from an arbitrary initial condition is very strong and referred to as minimality. Observe that for the flow to be minimal it suffices that, starting at $x(0) = 0$, it gets arbitrarily close to any point. Obviously, minimality is sufficient but not necessary for attainability, as the latter only requires the flow to pass arbitrarily close to a specific point on the torus.

Also observe that minimality makes existence of a $t$-solution to (9) independent of the $s_k$’s, hence independent of the states $i, j$ that exchange excitation. Then there exists a unique time $t$ at which $p_{\text{max}}(i,j)$ is reached for any pair $(i,j)$. Note that this attainability condition, which we shall refer to as simultaneous attainability, is actually a stronger condition than what we require for attainability above, which merely requires that a given precision can be achieved for all $i, j$ at some times $t_{i,j}$ but the $t_{i,j}$ could be different for different pairs $(i,j)$. 

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If there are at least three non-dark eigenspaces corresponding to $s_d, s_m, s_n \in \{\pm 1\}$, then there must exist a pair, say $(m, n)$, so that $s_m - s_n = 0$. In this situation, the $(m, n)$ equation (9) can be made to hold exactly whenever $\omega_{m,n} t = 2 \tau$ for $\tau \in \mathbb{N}$ and the remaining attainability equations for $(k, \ell) \neq (m, n)$, $\tau \in \mathbb{N}$, read $\theta_{k,\ell} \tau = \frac{1}{2} (s_k - s_\ell) \mod 2$ for $\theta_{k,\ell} = 2 \omega_{k,\ell}/\omega_{m,n}$. The left-hand side $\theta_{k,\ell} \tau$ of the preceding equation is the solution of the translation on the torus, that is, $x(\tau + 1) = x(\tau) + \theta \mod 2$ with initial condition $x(0) = 0$. The significant result [10, Prop. 1.4.1] is that the translation on the torus can come arbitrarily close to any point iff the elements in the set $(\theta_{1,2}, \theta_{2,3}, \ldots, \theta_{\tilde{N}-2,\tilde{N}-1}, 1)$ are linearly independent over $\mathbb{Q}$. As before, the linear independence is sufficient but not necessary for attainability.

**Application to Rings.** For uniform rings, we have specifically

$$
\omega_{k,\ell} = \frac{1}{\pi} (\lambda_k - \lambda_\ell) = -\frac{4}{\pi} \sin\left(\frac{\pi}{N} (k + \ell)\right) \sin\left(\frac{\pi}{N} (k - \ell)\right). \tag{10}
$$

Obviously, the elements of $\{\omega_{k,\ell} : k + \ell = 1, 3, \ldots, 2\tilde{N} - 3\}$ are not linearly independent over $\mathbb{Q}$ as it is easily verified that $\omega_{k',\ell'} = -\omega_{k,\ell}$ for $k' + \ell' = 2\tilde{N} - (k + \ell)$ but since all equations (7) are consistent, it suffices to remove either the $(k, \ell)$-equation or $(k', \ell')$-equation and eliminate such a lack of independence. Another lack of independence occurs when eigenvalues come in pairs (Lemma 1). However, since all equations (9) are again compatible, it suffices to retain one of the eigenvalues in every pair. Thus we have $t = 2 \tau/\omega_{m,n}$ for some pair $(m, n)$ and $\theta_{k,\ell} = 2 \sin\left(\frac{\pi}{N} (k + \ell)\right) / \sin\left(\frac{\pi}{N} (m + n)\right)$. Choosing $(m, n) = (0, 1)$ yields the attainability equations $\tau 2 \sin\left(\frac{\pi}{N} (k + \ell)\right) / \sin\left(\frac{\pi}{N}\right) = \frac{1}{2} (s_k - s_\ell) \mod 2$ for $(k, \ell) = (1, 2), (2, 3), \ldots, (\tilde{N} - 2, \tilde{N} - 1)$ and $\tau \in \mathbb{N}$.

**Example 1:** Consider a ring made up of $N = 5$ spins. The number of pairwise distinct eigenvalues of the single excitation Hamiltonian $\tilde{H}$ is $\tilde{N} = 3$ and there are two linearly independent transition frequencies $\omega_{0,1} = -\frac{4}{\pi} \sin\left(\frac{1}{5} \pi\right) \sin\left(\frac{4}{5} \pi\right) \sin\left(\frac{3}{5} \pi\right)$ and $\omega_{1,2} = -\frac{4}{\pi} \sin\left(\frac{3}{5} \pi\right) \sin\left(\frac{4}{5} \pi\right) \sin\left(\frac{1}{5} \pi\right)$. Hence, we must verify linear independence over $\mathbb{Q}$ of $\{\sin\left(\frac{1}{5} \pi\right), \sin\left(\frac{3}{5} \pi\right)\}$. A simple Mathematica exercise shows that

$$
\sin\left(\frac{\pi}{5}\right) = \frac{1}{4} \sqrt{10 - 2\sqrt{5}}, \quad \sin\left(\frac{3\pi}{5}\right) = \frac{1}{4} \sqrt{10 + 2\sqrt{5}}.
$$

Thus this entails checking that the only solution to

$$
\alpha_1 \frac{1}{4} \sqrt{10 - 2\sqrt{5}} + \alpha_3 \frac{1}{4} \sqrt{10 + 2\sqrt{5}} = 0, \quad \alpha_1, \alpha_3 \in \mathbb{Q},
$$

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is $\alpha = (\alpha_1, \alpha_3) = 0$. The above yields $\alpha_2^2(10 - 2\sqrt{5}) = \alpha_3^2(10 + 2\sqrt{5})$. Viewing the extension field $\mathbb{Q}(\sqrt{5})$ as a two-dimensional vector space over $\mathbb{Q}$ with basis $1, \sqrt{5}$ yields $\alpha_1^2 = \alpha_3^2$ and $\alpha_2^2 = -\alpha_3^2$, that is, $\alpha = 0$. Thus the flow on the torus is minimal and there exists a time such that $p_{\text{max}}(i,j)$ is attainable for all $(i,j)$.

**Example 2:** For $N = 10$ there are $\tilde{N} = 6$ distinct eigenvalues and five primary transition frequencies $\omega_{01}$, $\omega_{12}$, $\omega_{23}$, $\omega_{34}$, and $\omega_{45}$. It is easy to verify, e.g., using Mathematica that

$$\sin \left( \frac{\pi}{10} \right) = \sin \left( \frac{9\pi}{10} \right) = \frac{1}{4}(-1 + \sqrt{5}),$$
$$\sin \left( \frac{3\pi}{10} \right) = \sin \left( \frac{7\pi}{10} \right) = \frac{1}{4}(1 + \sqrt{5}),$$
$$\sin \left( \frac{5\pi}{10} \right) = 1,$$

and it is easily seen that $\alpha = (2, -2, 1)$ is a $\mathbb{Q}$-solution to the linear dependence equation

$$\alpha_1(-1 + \sqrt{5}) + \alpha_2(1 + \sqrt{5}) + 4\alpha_3 = 0.$$

Hence, in this case the various $p_{\text{max}}(i,j)$ are not simultaneously attainable.

### C. Simultaneous Diophantine Approximation

Instead of checking rational independence of $\{\theta_{(k,\ell)=(1,2), (2,3), \ldots, (\tilde{N}-2,\tilde{N}-1), 1}\}$, a less conservative approach is to proceed, either analytically and computationally [11], [12], via the *simultaneous Diophantine approximation* [13]–[16]:

$$\left| \theta_{k,\ell} - \frac{p_{k,\ell}}{q} \right| \leq \frac{c}{q^{1+\epsilon}}, \quad \forall (k,l) = (1,2), (2,3), \ldots, (\tilde{N},\tilde{N}+1),$$

$$p_{(k,\ell)=(1,2), (2,3), \ldots, (\tilde{N},\tilde{N}+1), q \in \mathbb{N}, \quad \epsilon > 0,$$

where we have discarded $1$ as its rational approximation is completely trivial and set $\tilde{N} := \tilde{N} - 2$ to denote the number of remaining constraints. With $\tau = q$, the above yields

$$|\theta_{k,\ell}\tau - p_{k,\ell}| \leq \frac{c}{\tau^{1+\epsilon}}, \quad |\{(k,\ell)\}| = \tilde{N}.$$

By the celebrated Dirichlet theorem that applies for $c = 1$ and $\epsilon = 1/\tilde{N}$, there are infinitely many integer solutions $q$ to the simultaneous Diophantine approximation; in other words, $\tau$ can become arbitrarily large, so that the above error can be made arbitrarily small.
Assuming we have obtained a Dirichlet-good simultaneous Diophantine approximation, the approximate attainability conditions become

\[ p_{k,\ell} = \frac{1}{2} (s_k - s_\ell) \mod 2, \quad (k, \ell) = (1, 2), \ldots, (\bar{N}, \bar{N} + 1). \]  

(11)

The difficulty is to find, if it exists, a simultaneous Diophantine approximation of Dirichlet accuracy that satisfies the above conditions on the numerators. The following example demonstrates that it is not, in general, possible to achieve the even/odd conditions (11) on the numerators \( p_{k,\ell} \) without compromising on the accuracy of the Diophantine approximation. To be more specific, arbitrary accuracy can still be achieved with Conditions (11), but a larger denominator is required to achieve the same level of accuracy as in the case where Conditions (11) are not enforced.

**Example 1 (Cont'):** We found that the flow on the torus is minimal and hence we can get arbitrarily close to an arbitrary point on the torus. By the preceding argument, this should guarantee existence of simultaneous Diophantine approximations of arbitrary accuracies with prescribed even/odd numerators depending on whether \( s_k - s_\ell = 0/\pm 1 \), respectively. Let us illustrate this for the \( N = 5 \) ring, for which the Diophantine approximation is \((\bar{N} = 1)\)-dimensional. It is readily found that

\[ \theta_{1,2} = \frac{2 \sin(3\pi/5)}{\sin(\pi/5)} = 1 + \sqrt{5} = [3; 4, 4, 4, 4, 4, \ldots], \]

where the final expression denotes the continued fraction expansion giving the optimal rational approximations [17, Chap. 10]. It is known that quadratic irrationality leads to continued fractions that eventually stabilize. The first convergents are

\[ 3, \frac{13}{4}, \frac{55}{17}, \frac{233}{72}, \frac{987}{305}, \frac{4181}{1292}, \frac{17711}{5473}, \frac{75025}{23184}, \frac{317811}{98209}, \frac{1346269}{416020}, \ldots \]

Observe that all of them have odd numerators, while the approximations we require must have even numerators since for the \( N = 5 \) ring one has \( s_0 = s_1 = 1 \). This can be rectified by using the so-called semi-convergents [18, Sec. V.4], [12]. Given two convergents \( p_{n-1}/q_{n-1} \) and \( p_n/q_n \) one can easily squeeze a semiconvergent between them as follows:

\[ \frac{p_{n-1}}{q_{n-1}} < \frac{p_{n-1} + p_n}{q_{n-1} + q_n} < \frac{p_n}{q_n}. \]

The semiconvergent has even numerator and has the accuracy of the convergents \( p_{n-1}/q_{n-1} \) and \( p_n/q_n \) but at the cost of doubling the denominator. Hence the error of using the semiconvergent \( |\theta_{0,1} - (p_{n-1} + p_n)| \leq c/\tau_{\text{semi}} \) for \( \tau_{\text{semi}} = q_{n-1} + q_n \) is multiplied by a factor of 2 relative to
the accuracy of using the convergent $|θ_{0,1}τ_{\text{convergent}} - p_n| < c/τ_{\text{convergent}}$ for $τ_{\text{convergent}} = q_n$. But the accuracy drop is bounded, hence one could still get arbitrary accuracy approximation with the semiconvergents provided $n$ is taken large enough.

**Remark 1:** The bound $c \geq 1$ can hardly be improved as for $c < 1$ there are “badly approximable vectors” $θ \in \mathbb{R}^N$ defined by $\liminf_{q \to \infty} q^{1/N} d(qθ, \mathbb{Z}^N) > 0$ and such that the simultaneous Diophantine approximation has only finitely many solutions [13], [19, Sec. 5]. If, however, $c$ is allowed to depend on $N$, refined bounds ($< 1$) can be derived on $c(N)$ for existence of infinitely many solutions [14]. Specializing the approximation to $N = 2$, it can be shown [20] that the bound can be improved down to $c = 8/13$, along with $\epsilon = 1/2$. In the 1-dimensional case Hurwitz’s theorem says that one can take $c = 1/\sqrt{5}$ and $\epsilon = 1$ although this is hardly of any significance in this problem.

**D. LLL-Algorithm**

The utility of the simultaneous Diophantine approximation is that it provides a sufficient condition for attainability that can be verified computationally. To simplify the notation, let $θ \in \mathbb{R}^N$, $p \in \mathbb{Z}^N$ be column-vectorizations of the $θ_{k,\ell}$’s, $p_{k,\ell}$’s, respectively, and rewrite the Diophantine approximation problem as

$$δ_q(θ) := \min_{p \in \mathbb{Z}^N} \|θq - p\|,$$

where the norm $\|\cdot\|$ can be taken either the sup-norm or the Euclidean norm.

$q$ is said to be best approximation denominator if $δ_r(θ) > δ_q(θ)$ for $r = 1, 2, \ldots, q - 1$. If $q$ is a best denominator, the solution $p$ to the above together with $q$ is called best approximation. The growth rate [15], [21] quantifies how fast a sequence of successive best approximation denominators $q_1 = 1 < q_2 < q_3 < \ldots$ has to grow to observe a decrease as $δ_{q_1}(θ) > δ_{q_2}(θ) > \ldots$. This rate is known to be geometric [21, Theorem 1.1], that is, for either the sup-norm or the Euclidean norm, $\liminf_{k \to \infty} q_k^{1/k} \geq 1 + 1/2N + 1$. As observed in Example 1 (Cont’), fixing the numerators $p_{k,\ell}$ to satisfy Conditions (11) can negatively affect the growth rate.

The first computational solution to the simultaneous Diophantine approximation was Lagarias’ geodesic multi-dimensional continued fraction expansion [13], [15], [22]. More recently, Lenstra, Lenstra and Lovász proposed an alternative algorithm, the so-called LLL algorithm [12], [15]. Both approaches proceed by reduction of the basis of the lattice generated by the columns of
$B(s) = \begin{pmatrix} I_{\bar{N} \times \bar{N}} & -\theta \\ 0_{1 \times \bar{N}} & s \end{pmatrix}$, where $s \downarrow 0$ is a scaling parameter. However, in order to broaden the set of approximations with the objective of capturing, if there exists, an approximation of acceptable accuracy that satisfies the even/odd conditions, we introduce nonuniform diagonal scaling and work on the lattice $\Lambda(s, X)$ generated by the columns of

$$B(s, X) = \begin{pmatrix} X & -X\theta \\ 0_{1 \times \bar{N}} & s \end{pmatrix},$$

where $X = \text{diag}(x_1, \ldots, x_{\bar{N}})$. Note that for $s = 1$ and $X = xI_{\bar{N} \times \bar{N}}$, we recover the scaling of [12]. There are many ways to reduce the basis of the lattice $B(s, X)\mathbb{Z}^{\bar{N} + 1}$, but we shall focus on the LLL-algorithm, which produces a basis of short Euclidean norm vectors

$$(b^*(s, X)_1, b^*(s, X)_2, \ldots, b^*(s, X)_{\bar{N} + 1}) =: B^*(s, X)$$

such that

$$\|b^*(s, X)_1\| < \|b^*(s, X)_j\|, \quad j = 2, \ldots, \bar{N} + 1.$$

The LLL-algorithm provides a $b^*(s, X)_1$ vector very close to the shortest one. A refined version of the LLL-algorithm captures the genuinely shortest vector of the lattice $\Lambda(s, X)$ as follows: Given the reduced basis $\{b^*(s, X)_i : i = 1, \ldots, \bar{N} + 1\}$, it turns out that the shortest (in the sense of the Euclidean norm) lattice vector is to be sought among all lattice vectors of the form $\sum_i \beta_i b^*(s, X)_i$, $|\beta_i| \leq (2/\sqrt{3})^{\bar{N} + 1}$. Lagarias’ theorem [15, Lemma 5] then implies that a shortest Euclidean norm vector of the lattice is a best $X$-weighted Diophantine approximation.

To see this, observe that

$$B(s, X) \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} X(p - \theta q) \\ sq \end{pmatrix}.$$  

Clearly, as $s \downarrow 0$, a shortest vector in the lattice $B(s, X)\mathbb{Z}^{\bar{N} + 1}$ provides a best $X$-weighted Diophantine approximation. With a short vector of $\Lambda(s, X)$, we construct a good Diophantine approximation as follows:

$$q = (B^*(s, X))_{\bar{N} + 1, 1} / s, \quad p_i = (B^*(s, X))_{1: \bar{N}, 1} / x_i + \theta_i q.$$  

With the shortest vector, we would construct the best approximation in the same way.

Unfortunately, but not unexpectedly, our simulations making use of the LLL-algorithm suggest that $q$ grows without bound as $s$ decreases to zero. With the “refined” LLL-algorithm, we can prove the following:
Theorem 1: The refined LLL-algorithm to solve

$$\left( \hat{p}(s), \hat{q}(s) \right) = \arg \min_{(p,q) \in \mathbb{Z}^{N+1}} \left\| B(s,X) \begin{pmatrix} p \\ q \end{pmatrix} \right\|_{\text{diag}(X,1)}$$

yields $\lim_{s \to 0} \hat{q}(s) = +\infty$.

Proof: Assume that there exist $s_{\min}$ and $q_{\max}$ such that, $\forall s \leq s_{\min}$, we have $q \leq q_{\max}$. Consider (13) for any $0 < s \leq s_{\min}$. By contradicting hypothesis, $\hat{q}(s) \leq q_{\max}$. The above yields a Diophantine approximation of $\theta$ but not the optimal one as $s \neq 0$. Now define

$$\left( \tilde{p}, \tilde{q} \right) = \arg \min_{(p,q) \in \mathbb{Z}^{N+1}} \left\| p - \theta q \right\|_X$$

along with

$$\delta(s) = \left\| \hat{p}(s) - \hat{q}(s)\theta \right\|_X^2 - \left\| \tilde{p} - \tilde{q}\theta \right\|_X^2.$$

Observe that there exists a lower bound $\delta_{\min}$ such that $\delta(s) \geq \delta_{\min} > 0$, since $\left\| \hat{p}(s) - \theta \hat{q}(s) \right\|_X$ cannot reach its minimum since $\hat{q}(s) \leq q_{\max}$. Now, consider the original problem (13) with $s < \min \left\{ \sqrt{\frac{\delta_{\min}}{2}}, s_{\min} \right\}$. With this choice, we have

$$\left( s\tilde{q} \right)^2 < \frac{\delta_{\min}}{2} < \frac{\delta(s)}{2} < \frac{\delta(s)}{2} + \left( s\hat{q}(s) \right)^2.$$

Then we have

$$\left\| \begin{pmatrix} \tilde{p} - \theta \tilde{q} \\ \tilde{q} \end{pmatrix} \right\|_{\text{diag}(X,1)}^2 = \left\| \tilde{p} - \theta \tilde{q} \right\|_X^2 + \left( s\tilde{q} \right)^2 \leq \left\| \hat{p}(s) - \theta \hat{q}(s) \right\|_X^2 - \delta(s) + \left( s\hat{q}(s) \right)^2 + \frac{\delta(s)}{2} = \left\| \hat{p}(s) - \theta \hat{q}(s) \right\|_X^2 + \left( s\hat{q}(s) \right)^2 - \frac{\delta(s)}{2}$$

$$= \left\| \begin{pmatrix} \hat{p}(s) - \theta \hat{q}(s) \\ s\hat{q}(s) \end{pmatrix} \right\|_{\text{diag}(X,1)}^2 - \frac{\delta(s)}{2} = \left\| \begin{pmatrix} \hat{p}(s) - \theta \hat{q}(s) \\ s\hat{q}(s) \end{pmatrix} \right\|_{\text{diag}(X,1)}^2 - \frac{\delta_{\min}}{2}.$$
Example 3: To illustrate the $X$-weighted algorithm, consider a homogeneous $N = 9$ ring along with $p_{\text{max}}(1, 4)$. It is readily found that

$$\theta = \begin{pmatrix} 5.0642 & 5.7588 & 3.7588 \end{pmatrix}^T$$

with constraints

$$\begin{pmatrix} \text{even} & \text{odd} & \text{odd} \end{pmatrix}^T.$$

Running the classical $X = \text{diag}\{1, 1, 1\}$ and the weighted $X = \text{diag}\{10, 10, 0.1\}$ algorithm on the simultaneous Diophantine approximation of $\theta$ yields the results shown in Fig. 1(b) and Fig. 1(b), respectively. The software creates a code 0 if the even/odd constraints are satisfied and a code 1 otherwise. The even/odd error code is shown in the top plots. Note that the even/odd constraints are never satisfied with the classical LLL-algorithm, but are satisfied for the weighted algorithm, first in a narrow band $s = [10^{-6.3}, 10^{-6.2}]$ and then for $s \leq 10^{-9.3}$. Also note that the approximation error is smaller with the weighted algorithm than with the unweighted algorithm for roughly the same denominator $q$ in the $s$-region where the even/odd constraints are satisfied.

As a general remark, the discrepancy among the diagonal entries of $X$ must be of an order of magnitude, as those entries must pass through “critical values” to make a difference.
E. Estimate of Time to Attain Maximum Probability

Our objective is to find an upper bound on the amount of time $t$ it takes to achieve $p_{\text{max}}(i, j) - p_t(i, j) \leq \epsilon_{\text{prob}}$. The approach is to translate the specification on the probability to a specification on the simultaneous Diophantine approximation. Proceeding from (3), recalling that $\text{Sgn}(\langle i|\Pi_k|j \rangle) = s_k = e^{-i\pi(s_k-1)/2 - 2\pi n_k}$, where $n_k$ is some integer, we obtain

$$\sqrt{p_{\text{max}}(i, j)} = \left| \sum_{k=0}^{N-1} \langle i|\Pi_k|j \rangle s_k \right|$$

$$= \left| \sum_{k \in K'} \langle i|\Pi_k|j \rangle e^{-i\frac{\pi}{2}(s_k - 1) - 2\pi n_k} \right|$$

$$= \left| \sum_{k \in K'} \langle i|\Pi_k|j \rangle e^{-i\frac{\pi}{2}(s_k - 1) - 2\pi n_k} e^{i\frac{\pi}{2}(s_l - 1) + 2\pi n_l} \right|$$

$$= \left| \sum_{k \in K'} \langle i|\Pi_k|j \rangle e^{-i\frac{\pi}{2}(s_k - s_l) - 2\pi n_k(n_k - n_l)} \right| .$$

In the second equation, the sum over $k$ has been replaced by a sum over $k \in K'$, since states such that $\langle i|\Pi_k|j \rangle = 0$ do not contribute to the sum. The third equality stems from the fact that $e^{i\frac{\pi}{2}(s_l - 1) + 2\pi n_l}$ is a global phase factor that is absorbed by the absolute value.

Next, we introduce the attainability condition (7), which is only approximately satisfied using the simultaneous Diophantine approximation. Exposing the gap between the left-hand side and the right-hand side of (7) yields

$$\sqrt{p_{\text{max}}(i, j)} = \left| \sum_{k \in K'} \left( \langle i|\Pi_k|j \rangle e^{-i(\lambda_k - \lambda_l)t} + \langle i|\Pi_k|j \rangle \left( e^{-i\frac{\pi}{2}(s_k - s_l) - 2\pi n_k(n_k - n_l)} - e^{-i(\lambda_k - \lambda_l)t} \right) \right) \right|$$

$$\leq \left| \sum_{k \in K'} \langle i|\Pi_k|j \rangle e^{-i(\lambda_k - \lambda_l)t} + \sum_{k \in K'} \langle i|\Pi_k|j \rangle \left( e^{-i\frac{\pi}{2}(s_k - s_l) - 2\pi n_k(n_k - n_l)} - e^{-i(\lambda_k - \lambda_l)t} \right) \right|$$

$$= \left| \sum_{k \in K'} \langle i|\Pi_k|j \rangle e^{-i\lambda_k t} + \sum_{k \in K'} \langle i|\Pi_k|j \rangle \left( e^{-i\frac{\pi}{2}(s_k - s_l) - 2\pi n_k(n_k - n_l)} - e^{-i(\lambda_k - \lambda_l)t} \right) \right|$$

$$\leq \sqrt{p_t(i, j)} + \sum_{k \in K'} \left| e^{-i\frac{\pi}{2}(s_k - s_l) - 2\pi n_k(n_k - n_l)} - e^{-i(\lambda_k - \lambda_l)t} \right| .$$

It follows that $\sqrt{p_{\text{max}}(i, j)} - \sqrt{p_t(i, j)} \leq \sum_{k \in K'} \left| e^{-i\frac{\pi}{2}(s_k - s_l) - 2\pi n_k(n_k - n_l)} - e^{-i(\lambda_k - \lambda_l)t} \right|$. The trivial identity

$$p_{\text{max}} - p_t = \left( \sqrt{p_{\text{max}}} - \sqrt{p_t} \right) \left( \sqrt{p_{\text{max}}} + \sqrt{p_t} \right)$$
then shows that to secure $p_{\text{max}}(i,j) - p_t(i,j) \leq \epsilon_{\text{prob}}$, it suffices to make
\[
\sum_{k \in K'} \left| e^{-i \frac{\pi}{2} (s_k - s_\ell)} - e^{-i(\lambda_k - \lambda_\ell)t} \right| \leq \frac{\epsilon_{\text{prob}}}{2},
\] (14)
where it is observed that $\ell \in K'$ is arbitrary.

The last step is to relate the left-hand side of (14) to the simultaneous Diophantine approximation error. Define $\epsilon_{\text{Da}}(m, m+1) := |\theta_{m,m+1} q - p_{m,m+1}|$ and $\|\epsilon_{\text{Da}}\|_\infty = \max_{1 \leq m \leq \bar{N}} \epsilon_{\text{Da}}(m, m+1)$. We have
\[
\sum_{k \in K'} \left| e^{-i \frac{\pi}{2} (s_k - s_\ell)} - e^{-i(\lambda_k - \lambda_\ell)t} \right| \leq 2 |K'| \max_{k \in K'} \sin \left( \frac{\pi}{2} \sum_{m=k}^{\ell-1} \epsilon_{\text{Da}}(m, m+1) \right)
\leq 2 |K'| \sin \left( \frac{\pi}{2} \epsilon_{\text{Da}} \|\epsilon_{\text{Da}}\|_\infty \right).
\]

A few words of explanation: The second equality assumes that $k \leq \ell$. The case of the opposite inequality is treated similarly and omitted. The third equality is valid when $t$ is chosen so as to have the $(0,1)$ attainability constraint satisfied exactly. The second inequality comes from $\ell - 1 \leq \bar{N} = \tilde{N} - 2$ together with the hypothesis that the Diophantine approximation is tight enough to guarantee $\sum_{m=1}^{\tilde{N}-2} \epsilon_{\text{Da}}(m, m+1) \leq 1$. Likewise, the third and last inequality tacitly assumes that $(\bar{N}) \|\epsilon_{\text{Da}}\|_\infty \leq 1$. From the above, it follows that for the attainability accuracy $\epsilon_{\text{prob}}$ to be reached, it is sufficient to take
\[
2 |K'| \sin \left( \frac{\pi}{2} (\bar{N}) \|\epsilon_{\text{Da}}\|_\infty \right) < \frac{\epsilon_{\text{prob}}}{2}.
\]

There are many simultaneous Diophantine approximation schemes, but if we retain the Dirichlet
one, \( \| \epsilon_{D_a} \|_\infty \leq 1/q^{1/N} \), we obtain the further sufficient condition

\[
2|K'| \sin \left( \frac{\pi}{2} \left( \bar{N} \frac{1}{q^{1/N}} \right) \right) < \frac{\epsilon_{\text{prob}}}{2}.
\]

A minimum \( q \) that guarantees \( \epsilon_{\text{prob}} \) is easily extracted from the above inequality as

\[
q \geq \left( \frac{\pi \bar{N}}{2 \sin^{-1} \left( \frac{\epsilon_{\text{prob}}}{4|K'|} \right)} \right)^N.
\]

The latter provides us with the amount of discrete-time steps \( \tau = q \) it takes to reach an attainability accuracy of \( \epsilon_{\text{prob}} \). Contrasting the above with numerical simulation of (3) reveals that the bound \( 0 \left( \bar{N}^N \right) \) is conservative, but as easily seen it is a very general bound. All that needs to be generalized to a particular spintronic network is \( \lambda_k, s_k \). From any general \( (1/\pi)(\lambda_k - \lambda_{m+1})t - (1/2)(s_m - s_{m+1}) \) to be made small mod 2, the Dirichlet approximation would yield \( q = 0 \left( \bar{N}^N \right) \).

Given the number of discrete-time steps \( q \), the amount of continuous time \( t \) in \( \hbar = 1 \) units necessary to reach an attainability accuracy of \( \epsilon_{\text{prob}} \) is given by the general formula \( t = 2q/\omega_{m,n} \), where the specific form of \( \omega_{m,n} \) depends on the network structure.

F. Computational Implementation

For practical computation of the time steps \( \tau = q \), we must find nominators \( p_{k,\ell} \) that fulfill the odd/even constraints using the LLL-algorithm. The nonuniform variant introduced above makes it simpler to find suitable parameters \( X \) and \( s \), but a search is still required. To automate the search, rather doing it “by hands” as in Example 3, we use a standard genetic algorithm that tries to find \( X \) with a user-defined \( s \) by minimizing the number of even/odd constraint violations of the \( p_{k,\ell} \)'s. This works well in most cases, requiring only a few iterations (typically up to 5) for reasonably sized populations (about 200). We suggest that the standard crossover and mutation operators could be adjusted to improve the performance of the search. In particular, increasing the likelihood of changing the \( X \) values corresponding to those nominators \( p_{k,\ell} \) which violate a constraint, and increasing the likelihood of retaining those \( X \) values for which the corresponding \( p_{k,\ell} \) do not violate the constraints may improve performance.
IV. INFORMATION TRANSFER (IN-)FIDELITY METRIC AND GEOMETRY

A. Definition and Motivation of ITF Prametric

To develop a geometric picture, we can view a spintronic network as a pre-metric or more precisely a prametric space\(^2\) endowed with the prametric that quantifies the Information Transfer Infidelity (ITI). To fix terminology, recall that given a graph \(G = (V, E)\), or any set of points \(V\) for that matter, a prametric [24, p. 666], [23, p. 23] is a function \(d : V \times V \to \mathbb{R}_{\geq 0}\) such that (i) \(d(i, j) \geq 0\) and \(d(i, i) = 0\).

To derive a suitable prametric on the vertex set \(V = \{|i\} : i = 1, \ldots, N\}\) from the probability \(p_{\text{max}}\) data, we inspire ourselves from a similar situation in sensor networks [25], where \(V\) is the set of sensors and a Packet Reception Rate \(\text{PRR}(i, j)\) is defined as the probability of successful transmission of the packets from sensor \#i to sensor \#j. After symmetrization of the packet reception rate, a prametric (in fact, a semi-metric [26]–[28]) can be defined as \(d(i, j) = -\log \text{PRR}(i, j)\).

Should there be a violation of the triangle inequality, say, \(d(i, j) > d(i, k) + d(k, j)\), then the distance between \(i\) and \(j\) is redefined as \(d(i, k) + d(k, j)\). The importance of the metric is that it provides a notion of network curvature, which has a dramatic impact on the traffic flow [29], [30] in a paradigm that extends to quantum chains [31]. Following sensor network intuition [25], we define

\[
d(i, j) = -\log p_{\text{max}}(i, j). \tag{15}
\]

Obviously, \(d(i, j) \geq 0\) and, as it will be shown in Th. 2, it is not hard to prove that \(d(i, i) = 0\).

Observe that we could define the time-stamped prametric by \(d_t(i, j) = -\log p_t(|i\rangle, |j\rangle)\) except that in general \(d_t(i, i) \neq 0\). To remedy this situation, we could define \(d(i, j) = \inf_{t \geq 0} d_t(i, j) = -\log \sup_{t \geq 0} p_t(i, j)\). Since, by Cauchy-Schwarz, \(p_t(i, i) \leq 1\) and \(p_{t=0}(i, i) = 1\), we have \(\sup_{t \geq 0} p_t(i, i) = 1\) and hence \(d(i, i) = 0\). This alternate prametric definition is equivalent to the earlier one when \(p_{\text{max}}\) is attainable, but it reveals that this prametric makes the network of finite diameter (\(\sup_{i,j} d(i, j) < \infty\)) as \(N \to \infty\) as Theorem 2 will show. This has the unfortunate consequence of preventing a genuine large-scale analysis. As Section V will show a bias rectifies this problem (see also [31]).

\(^2\)We prefer to avoid the terminology of pre-metric space since it is not quite accepted; prametric on the other hand is the terminology introduced by Arkhangel’skii and Pontryagin [23].
Generally, this information transfer infidelity prametric is not a proper distance satisfying the triangle inequality, but for certain networks such as rings with uniform coupling this prametric will be shown to define a proper distance.

This quantum mechanical (pra)metric is quite different from the usual Euclidean distance $d_E$ of the spins in the spintronic device. In particular, two spins that are physically close in the medium may be far quantum mechanically, and conversely. If two spins are quantum mechanically far, control is necessary to enable transmissions that were too weak or forbidden by the natural quantum mechanical couplings. This control of information can be viewed as the problem of controlling the quantum mechanical geometry of the network.

B. ITF Distance Geometry of Uniform Spin Rings

It could be argued that a prametric is sufficient if we are solely interested in assessing the difficulty of communication or fidelity of information transfer between nodes in a network. However, a proper metric allows us to investigate other geometric properties such as the curvature of the network with regard to the ITF.

A prametric $d : V \times V \to \mathbb{R}_{\geq 0}$ is a pseudo-metric if in addition to (i) $d(i, j) \geq 0$, $d(i, i) = 0$, it satisfies (ii) $d(i, j) = d(j, i)$ and (iii) the triangle inequality $(d(i, j) \leq d(i, k) + d(k, j))$ holds. A metric or distance is a pseudo-metric that has (iv) the separation property: $d(i, j) = 0$ only if $i = j$.

**Theorem 2:** For a quantum ring $(V_N, E_N)$ of $N$ uniformly distributed spins with XX or Heisenberg couplings, $d_N(i, j) := -\log p_{\max}(i, j)$ has the following properties:

1) For $N$ odd, $(V_N, d_N)$ is a metric space.
2) For $N$ even, $(V_N, d_N)$ is a pseudo-metric space that becomes metric after antipodal point identification.
3) If $N = p$ or $N = 2p$, where $p$ is a prime number, then the distances on the space of equivalence classes of spins are uniform, i.e., $d_N(i, j) = c_N$ for $i \neq j$. Otherwise, the distances are non-uniform.
4) In all cases $\lim_{N \to \infty} d_N(i, j) = 2 \log \frac{\pi}{2}$, $i \neq j \mod (N/2)$.

**Proof:** To show that $(V_N, d_N)$ is a pseudo-metric space we need to verify that (i) $d_N(i, i) = 0$, (ii) $d_N(i, j) = d_N(j, i)$, and (iii) the triangle inequality holds. For a metric space we must further have (iv) $d_N(i, j) \neq 0$ unless $i = j$. 
(i) is clearly satisfied as the projectors onto the eigenspaces are a resolution of the identity, \( \sum_k \Pi_k = I \), and thus for any unit vector \(|i\rangle\), we have \( \sum_{k=1}^N |\langle i|\Pi_k|i\rangle| = \sum_{k=1}^N |\Pi_k|i\rangle|^2 = 1 \).

(ii) follows from \( |\langle i|\Pi_k|j\rangle| = |\langle j|\Pi_k|i\rangle| \). The proof of the remaining properties relies on the circulant matrix property of the Hamiltonian \( \bar{H} \) in the single excitation subspace \( \bar{H} \), as shown in Eq. (2) and Table I.

Observe in Table I the double eigenvalues \( \lambda_k = \lambda_{N-k} \), except for \( k = 0 \) and \( k = N/2 \) if \( N \) even. From Table I, each of these double eigenvalues has two general complex conjugate eigenvectors. These general eigenvectors need not be orthogonal, but observing that \( \langle w_k|w_{k}\rangle = \delta_{k\ell} \) and \( \langle w_k|w^*_k\rangle = 0 \) shows that

\[
|v_0\rangle = |w_0\rangle = \frac{1}{\sqrt{N}}(1, 1, \ldots)^T, \\
|v_k\rangle = |w_k\rangle, \quad |v_{N-k}\rangle = |w^*_k\rangle, \quad k = 1, \ldots, N' = \left\lfloor \frac{N-1}{2} \right\rfloor, \\
|v_{N/2}\rangle = |w_{N/2}\rangle = \frac{1}{\sqrt{N}}(1, -1, \ldots)^T, \quad \text{if } N \text{ is even},
\]

defines an orthonormal basis of \( \mathcal{H} \). Furthermore, in the basis in which \( \bar{H} \) is circulant, we have \( |i\rangle = e_i \), where \( \{e_i : i = 1, \ldots, N\} \) is the natural basis of \( \mathbb{C}^N \). We have

\[
|\langle i|\Pi_0|j\rangle| = |\langle i|v_0\rangle\langle v_0|j\rangle| = \frac{1}{N}, \\
|\langle i|\Pi_k|j\rangle| = |\langle i|v_k\rangle\langle v_k|j\rangle + \langle i|v_{N-k}\rangle\langle v_{N-k}|j\rangle| \\
= |\rho_{ki}^N(\rho_{kj}^N)^* + (\rho_{ki}^N)^*\rho_{kj}^N|\frac{1}{N} \\
= |\rho_{ki}^{N(i-j)} + \rho_{kj}^{-N(i-j)}|\frac{1}{N} = \frac{2}{N}|\cos\left(\frac{2\pi k(i-j)}{N}\right)|, \\
|\langle i|\Pi_{N/2}|j\rangle| = |\langle i|v_{N/2}\rangle\langle v_{N/2}|j\rangle| = \frac{1}{N}.
\]

Summing over all eigenspaces \( k = 0, \ldots, \lfloor N/2 \rfloor \) gives

\[
\sqrt{p_{\text{max}}(i,j)} = \begin{cases} \\
\frac{1}{N} + \frac{2}{N} \sum_{k=1}^{N'} \cos\left(\frac{2\pi k(i-j)}{N}\right), & \text{if } N = 2N' + 1, \\
\frac{2}{N} + \frac{2}{N} \sum_{k=1}^{N'} \cos\left(\frac{2\pi k(i-j)}{N}\right), & \text{if } N = 2N' + 2.
\end{cases}
\]

For \( N = 2N' + 1 \), it is easy to see that \( p_{\text{max}}(i,j) = 1 \) if and only if \( i = j \), hence (iv). For \( N = 2N' + 2 \), on the other hand, we also have \( |\cos\left(\frac{2\pi kN/2}{N}\right)| = |\cos(\pi k)| = 1 \), and thus \( d(i,j) = 0 \) for \( i - j = N/2 \), i.e., the distance vanishes for antipodal points, and thus \( d(i,j) \) is at most a pseudo-metric. However, noting that \( d(i,j) = d(i, N' + 1 + j) \) for \( j \leq N' + 1 \), we can
identify antipodal points $|j\rangle$ and $|j+N'+1\rangle$ and let $d$ be defined on the set of equivalence classes $[[j]]$ for $j = 1, \ldots, N'+1$. (Note that antipodal identification preserves the ring structure.) At this stage, $d$ is a semi-metric [26]–[28], that is, it satisfies all axioms of a metric except the triangle inequality.

To show that the triangle inequality is satisfied, we show that $\sqrt{p_{\text{max}}(i,m)} \sqrt{p_{\text{max}}(m,j)} \leq \sqrt{p_{\text{max}}(i,j)}$. From the definition (3) of $p_{\text{max}}$ rewritten using (17)-(19) in terms of the eigenvectors of $\bar{H}$ we have

$$\sqrt{p_{\text{max}}(i,m)} = \frac{1}{N} \sum_{k=0}^{N-1} \alpha_k \rho_N^{k(m-i)},$$

$$\sqrt{p_{\text{max}}(m,j)} = \frac{1}{N} \sum_{k'=0}^{N-1} \beta_{k'} \rho_N^{k'(j-m)},$$

where $\alpha_k = s_k(i,m) = \text{Sgn} \left( \rho_N^{k(m-i)} + \rho_N^{-k(m-i)} \right) \in \{\pm 1, 0\}$ is rewritten explicitly in terms of the eigenvectors rather than as in Sec. III and $\beta_{k'} = s_{k'}(m,j)$. Setting

$$\gamma_k = \sum_{k'=0}^{N-1} \alpha_k \beta_{k'} \rho_N^{(k'-k)(j-m)}$$

we obtain

$$\sqrt{p_{\text{max}}(i,m)} \sqrt{p_{\text{max}}(m,j)} = \frac{1}{N^2} \sum_{k,k'=0}^{N-1} \alpha_k \beta_{k'} \rho_N^{k(m-i)} \rho_N^{k'(j-m)}$$

$$= \frac{1}{N^2} \sum_{k,k'=0}^{N-1} \alpha_k \beta_{k'} \rho_N^{k(j-i)+(k'-k)(j-m)}$$

$$= \frac{1}{N^2} \sum_{k=0}^{N-1} \gamma_k \rho_N^{k(j-i)} = \left| 1 \sum_{k=0}^{N-1} \gamma_k \rho_N^{k(j-i)} \right| .$$

The final equality follows because the LHS and thus the RHS are known to be real and positive. Furthermore, as $\rho_N$ is a root of unity, $|\rho_N| = 1$, and recalling $|\alpha_k| = |\beta_{k'}| = 1, 0$,

$$|\gamma_k| = \left| \rho_N^{k(m-j)} \sum_{k'=0}^{N-1} \alpha_k \beta_{k'} \rho_N^{k'(j-m)} \right|$$

$$\leq \left| \rho_N^{k(m-j)} \right| \sum_{k'=0}^{N-1} \left| \alpha_k \beta_{k'} \rho_N^{k'(j-m)} \right| \leq N,$$
where the last inequality allows for the presence of dark states. Again we have $\rho^{(N-k)(m-j)}_N = \rho^{-(k(m-j)}_N$, and as the LHS above is known to be real, we know that we must have $\gamma_k = \gamma_{N-k}$.

Hence, we can again collect exponential terms pairwise to obtain cosines, which gives for $N = 2N' + 1$:

$$\left| \frac{1}{N^2} \sum_{k=0}^{N-1} \gamma_k \rho^k_{N} \right| = \left| \frac{\gamma_0}{N^2} + \frac{1}{N^2} \sum_{k=1}^{N'} 2\gamma_k \cos \left( \frac{2\pi k(j-i)}{N} \right) \right|$$

$$\leq \left| \frac{\gamma_0}{N^2} \right| + \frac{2}{N^2} \sum_{k=1}^{N'} |\gamma_k| \left| \cos \left( \frac{2\pi k(j-i)}{N} \right) \right|$$

$$\leq \frac{1}{N} + \frac{2}{N} \sum_{k=1}^{N'} \left| \cos \left( \frac{2\pi k(j-i)}{N} \right) \right|$$

$$= \sqrt{p_{\text{max}}(i,j)}.$$

For $N = 2N' + 2$, we simply replace $\gamma_0$ by $\gamma_0 + \gamma_{N'+1}$ above to obtain

$$\left| \frac{1}{N^2} \sum_{k=0}^{N-1} \gamma_k \rho^k_{N} \right| \leq \frac{2}{N} + \frac{2}{N} \sum_{k=1}^{N'} \left| \cos \left( \frac{2\pi k(j-i)}{N} \right) \right|$$

$$= \sqrt{p_{\text{max}}(i,j)}.$$

This proves (iii) and hence parts (1) and (2) of the theorem.

To establish (3), we note that if $N = 2N' + 1$ is prime then

$$\sum_{k=1}^{N'} \left| \cos \left( \frac{2\pi k(j-i)}{N} \right) \right| = \sum_{k=1}^{N'} \left| \cos \left( \frac{2\pi k}{N} \right) \right|.$$

If $N$ is not $\pi$ or $2\pi$ then $N$ and $(i-j)$ will have factors (which can be canceled) in common for some $(i-j)$ but not for others and hence we will obtain different distances.

To establish (4), letting $N \to \infty$, it is easily seen that the dependency on $i, j$ is eliminated provided $i \neq j \mod (N/2)$. Hence, taking the norm of the above and then $-\log(\cdot)$ it follows that, at the infinite ring limit, the distance is uniform for $i \neq j + \mod(N/2)$. Finally,

$$\lim_{N \to \infty} \sqrt{p_{\text{max}}(i,j)} = \lim_{N \to \infty} \frac{2}{N} \sum_{k=0}^{N/2} |\cos((i-j)2\pi k/N)|$$

$$= \frac{2|i-j|}{\pi} \int_{0}^{2\pi |i-j|} \cos(|i-j| x) dx$$

$$= \frac{2|i-j|}{\pi |i-j|} \left[ \sin(|i-j| x) \right]_{0}^{2\pi |i-j|} = \frac{2}{\pi}.$$
shows that \( \lim_{N \to \infty} d_N(i,j) = 2 \log \frac{\pi}{2} \approx 2 \times 0.4516 \) for \( i \neq j \mod (N/2) \).

Case 3) of the preceding theorem allows for a very specific geometrization of the quantum ring in terms of constant curvature spaces. Define the \( n \)-sphere of curvature \( \kappa \) as \( S^n_{\kappa} := \{ x \in \mathbb{R}^{n+1} : \| x \|^2 = 1/\kappa \} \). We have the following corollary:

**Corollary 1:** The metric space \( (\mathcal{V}_p, d_p) \) of \( p \) spins \((p \geq 3 \text{ prime})\) arranged in a homogeneous ring with uniform ITI distance \( d_p(i,j) = c_p, i \neq j \), is isometrically embeddable in \( S^{p-1}_{\kappa} \) iff

\[
\kappa \leq \left[ \frac{1}{c_p} \cos^{-1} \left( -\frac{1}{p-1} \right) \right]^2.
\]  

Furthermore, it is irreducibly isometrically embeddable in \( S^{p-2}_{\kappa} \) for

\[
\kappa = \left[ \frac{1}{c_p} \cos^{-1} \left( -\frac{1}{p-1} \right) \right]^2.
\]  

**Notes:** In the above, “irreducibly embeddable” means that the embedding cannot happen into a lower-dimensional constant curvature space. By convention, \( \cos^{-1} \) takes values in \([\pi/2, \pi]\).

**Proof:** This result is a corollary of [32, Th. 63.1]. For the details, see [33, Appendix].

Note that this corollary deals with embeddability of the vertices only; however, edges can be mapped isometrically as arcs of great circles on either the sphere of curvature (21) or that of curvature (22). Also note that the symmetry of the simple \( p = 3 \) case of the circle \( S^1 \) circumscribed to a equilateral triangle is misleading, as in very high dimension \((p \to \infty)\), Eq. (22) yields \( R \to \frac{c_p}{\pi/2} \), that is, all vertices are mapped to the half-sphere of radius \( R \).

Regarding the \( \mathcal{V}_{2p} \) case, we could first do the anti-podal identification on the combinatorial ring \((\mathcal{V}_{2p}, \mathcal{E}_{2p})\), leading to a \((\mathcal{V}_p, \mathcal{E}_p)\) ring, and then embed \((\mathcal{V}_p, \mathcal{E}_p)\) as in the preceding corollary.

Regarding Case 4) when \( N \) is odd, define \( \epsilon := \max_{i \neq j} |d_N(i,j) - 2 \log(\pi/2)| \). Then the metric space \((\mathcal{V}_N, d_N)\) can be mapped isometrically into the sphere \( S^{N-2}_{\kappa} \) of radius \( d_\infty / \cos^{-1} \left( -\frac{1}{(N-1)} \right) \) up to an additive distortion not exceeding \( \epsilon \), that is, the embedding is quasi-isometric [34, 7.2.G]. The case of an even \( N \) is dealt with as before using anti-podal identification. The geometry of a genuinely infinite ring \((N = \infty \text{ rather than } N \to \infty)\) is completely different and is left to a further paper.

The \( N \) even case can be dealt with in a different way. Rather than doing, first, a combinatorial anti-podal identification \((i = j \text{ if } i - j = 0 \mod (N/2))\) and, then, mapping the quotient space \( \mathcal{V}_N / \sim \) to the sphere, we could map the combinatorial antipodal points to geometrical antipodal points on the sphere \( S^{N-2}_{\kappa} \) with the understanding that geometrical antipodal points on the
sphere are identified to yield the real projective space $\mathbb{R}P^{N-2}$. A slight generalization of (22) of Corollary 1 together with 4) of Theorem 2 yields an irreducible embedding of $(\mathcal{V}_N, d_N)$ into the sphere of curvature $\kappa = \left(\frac{\cos^{-1}\left(-\frac{1}{N-1}\right)}{2\log\frac{\pi}{2}}\right)^2$. On the other hand, $\mathbb{R}P^{N-2}$ is usually endowed with the standard curvature 1 metric of diameter $\pi/2$. To sum up,

*Corollary 2:* For $N$ even, there is an embedding $\mathcal{V}_N \hookrightarrow \mathbb{R}P^{N-2}$, which is quasi-isometric for the scaled distance $d_N \cos^{-1}\left(-\frac{1}{N-1}\right) / \left(4\log\frac{\pi}{2}\right)$ on $\mathcal{V}_N$ and the curvature 1 distance on $\mathbb{R}P^{N-2}$.

Furthermore, for $N \to \infty$ the distortion becomes vanishingly small.

V. CONTROL OF INFORMATION TRANSFER FIDELITY

To overcome intrinsic limitations on quantum state transfer or speed up transfer, one can either try to engineer spin chains or networks with non-uniform couplings or introduce dynamic control to change the network topology. The idea of engineered couplings was originally proposed to achieve perfect state transfer between the end spin in spin chain quantum wires [35]. Our analysis above shows that engineering the couplings is not strictly necessary. For an XX or Heisenberg-type chain with uniform nearest-neighbor couplings, for example, it can easily be verified that the information transfer fidelity between the end spins is unity, and attainability of the bounds means that we can achieve arbitrarily high state transfer fidelities between the end spins if we wait long enough. Engineering the couplings, however, can speed up certain state transfer tasks such as state transfer between the end spins at the expense of others.

A more flexible alternative to fixed engineered couplings is to apply *control* to change the network geometry and hence speed up state transfer as well as enable some transfers that either were forbidden or had poor ITF. One way this can be achieved is to apply static electromagnetic bias fields to change the energy-level splittings between the spin-up and spin-down states for different nodes in the graph. To see how the application of such bias fields can alter the transfer fidelities and network geometry, consider a simple, concrete example of a single bias field $\zeta$ applied to node $\ell$ in a spin ring with uniform coupling. First, due to translation invariance, we can always relabel the nodes so that the biased node is node $N$. Then, assuming XX coupling,
the Hamiltonian on the single excitation subspace becomes

\[
\tilde{H}_N^{(\zeta)} = \begin{pmatrix}
0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 & 1 \\
1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 1 & 0 & 1 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 1 \\
1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1 & \zeta \\
\end{pmatrix},
\]

where it is observed that we have the decomposition

\[
\tilde{H}_N^{(\zeta)} = C_N + \zeta E_{N,N},
\]

where \(C_N\) is the \(N \times N\) circulant matrix defined above and \(E_{N,N}\) is a \(N \times N\) matrix which is zero except for a 1 at position \((N, N)\).

Physically, applying a large bias field to the \(N\)th node in the ring results in a large detuning that effectively eliminates this node from the ring and breaks the ring open, leaving a chain of length \(N - 1\). Hence, in the limit \(\zeta \rightarrow \infty\), we expect the transition fidelities for the first \(N - 1\) nodes to approach those for a chain of length \(N - 1\) while the transition fidelities between the first \(N - 1\) nodes and the final (biased) node approach 0. We now reformulate this intuitively obvious result in precise mathematical language.

**Lemma 2:** The eigenvalues and eigenvectors of the \((N - 1) \times (N - 1)\) Toeplitz matrix \(T_{N-1}\) made up of ones on the super diagonal and subdiagonal and zeros everywhere else are given by \(\lambda_k = 2 \cos \left( \frac{\pi k}{N} \right)\) and \(|w_k\rangle_i = \sqrt{\frac{2}{N}} \sin \left( \frac{\pi ki}{N} \right); k = 1, \ldots, N - 1, i = 1, \ldots, N - 1\). Furthermore, for \(k\) even, \(|w_k\rangle_1 + |w_k\rangle_{N-1} = 0\).

**Theorem 3:** Let \(p_{\text{chain}}^{N-1}\) be the maximum transfer fidelities for a spin chain of length \(N - 1\) with uniform coupling between nearest neighbors. Let \(p_{\text{ring}}^{N,\zeta}\) be the maximum transfer fidelities
for a ring of size $N$ with bias $\zeta$ on the $N$th node. Then

$$\lim_{\zeta \to \infty} p_{\text{ring}}^{N,\zeta}(i,j) = \begin{cases} 
p_{\text{chain}}^{N-1}(i,j), & \text{if } i, j < N; \\
0, & i = N, j \neq N \text{ or } i \neq N, j = N; \\
1, & i, j = N. 
\end{cases} \quad (24)$$

**Proof:** Write the characteristic polynomial of $\bar{H}_{N}^{\zeta}$ as $\det((\lambda I_{N} - C_{N}) - \zeta E_{N,N})$ and recall that the determinant of the sum of two matrices equals the sums of the determinants of all matrices made up with some columns of one matrix and the complementary columns of the other matrix. Applying the latter to the characteristic polynomial of $\bar{H}_{N}^{(\zeta)}$ yields

$$\det(\lambda I_{N} - \bar{H}_{N}^{(\zeta)}) = \det(\lambda I_{N} - C_{N}) - \zeta \det(\lambda I_{N} - T_{N-1}),$$

where $T_{N-1}$ is the Toeplitz matrix defined in the lemma. From classical root-locus techniques, it follows that, as $\zeta \to \infty$, exactly one eigenvalue $\lambda_{N}(\zeta)$ goes to $\infty$, while the remaining ones $\lambda_{1}(\zeta), \ldots, \lambda_{N-1}(\zeta)$ converge to the roots of $\det(\lambda I_{N-1} - T_{N-1}) = 0$.

Next, we look at the eigenvectors and rewrite the eigenvector equation as

$$\begin{pmatrix}
T_{N-1} & 1 \\
0_{N-3} & 1 \\
1 & \zeta
\end{pmatrix}
\begin{pmatrix}
|w_{k}(\zeta)\rangle_{1} \\
|w_{k}(\zeta)\rangle_{N-1} \\
|w_{k}(\zeta)\rangle_{N}
\end{pmatrix} = \lambda_{k}(\zeta)
\begin{pmatrix}
|w_{k}(\zeta)\rangle_{1} \\
|w_{k}(\zeta)\rangle_{N-1} \\
|w_{k}(\zeta)\rangle_{N}
\end{pmatrix}.$$

Consider first the first $k \neq N$ equations. Since $\lim_{\zeta \to \infty} \lambda_{k}(\zeta)$ exists and is finite, it follows from the bottom eigenequation that $\zeta |w_{k}(\zeta)\rangle_{N}$ remains bounded as $\zeta \to \infty$. Therefore, $\lim_{\zeta \to \infty} |w_{k}\rangle_{N} = 0$. Since $\lambda_{k}(\infty)$ is a unique eigenvalue of $T_{N-1}$, it follows that $\lim_{\zeta \to \infty} |w_{k}(\zeta)\rangle_{1:N-1}$ is the corresponding eigenvector of $T_{N-1}$. It remains to show that with this $|w_{k}\rangle_{1:N-1}$ the bottom eigenequation can be made to hold. This is easily achieved by defining

$$\lim_{\zeta \to \infty} \zeta |w_{k}(\zeta)\rangle_{N} = -\lim_{\zeta \to \infty} (|w_{k}(\zeta)\rangle_{1} + |w_{k}(\zeta)\rangle_{N-1})$$

By the lemma, for $k$ even, we have $\lim_{\zeta \to \infty} \zeta |w_{k}(\zeta)\rangle_{N} = 0$, and therefore the $k < N$ eigenequation holds with $|w_{k}(\zeta)\rangle_{N}$ going to zero faster than $1/\zeta$. For $k$ odd, $|w_{k}(\zeta)\rangle_{N}$ goes to zero as $c/\zeta$, where $c \neq 0$ is some constant.
By the root locus result, for \( \zeta \) large enough, all eigenvalues are distinct, and we have
\[
\sqrt{p^{(\zeta, N)}_{\text{ring}}(i, j)} = \sum_{k<N} |\langle i|w_k(\zeta)\rangle\langle w_k(\zeta)|j\rangle| + |\langle i|w_N(\zeta)\rangle\langle w_N(\zeta)|j\rangle| = \sqrt{p^{(N-1)}_{\text{chain}}(i, j) + |\langle i|w_N(\zeta)\rangle\langle w_N(\zeta)|j\rangle|}
\]
(25)
where the second equality is understood as the \( \zeta \to \infty \) limit. To complete the proof, it therefore remains to look at \( |w_N(\zeta)\rangle \).

The top \( k = N \) eignequation easily implies that \( \zeta |w_N(\zeta)\rangle_{1:N-1} \) remains bounded as \( \zeta \to \infty \). Therefore \( \lim_{\zeta \to \infty} |w_N(\zeta)\rangle_{1:N-1} = 0 \). To normalize the eigenvector, we take \( \lim_{\zeta \to \infty} |w_N(\zeta)\rangle_N = 1 \). The latter together with (25) proves the theorem.

Thus we have a systematic way to compute the asymptotic transfer probability of a ring with high bias from the transfer probability of a chain without bias.

**Example 4:** As an illustration of how these results can be used, consider a ring of size \( N = 9 \). The maximum transfer fidelities between nodes \( i \neq j \) for this ring are quite low, 0.4094 and 0.4444, as shown by the light green and the dark green, resp., squares of Fig 2. However, applying a large bias to node 9 changes the maximum transfer fidelities. In particular the maximum transfer fidelity between nodes 1 and 8, 2 and 7, 3 and 6, and 4 and 5, for example, now approaches 1.

Fig. 2 shows a visual representation of the transfer fidelities for the ring without bias (left) and with bias (right). This result is fully consistent with Theorem 3, as using Lemma 2, it is easily verified that
\[
\sqrt{p^{(8)}_{\text{ring}}(i, 9-i)} = \frac{2}{9} \sum_{k=1}^{8} \sin \left( \frac{\pi ik}{9} \right) \sin \left( \frac{\pi (9-i)k}{9} \right) = 1.
\]

This example shows how a dynamic routing scheme can be implemented to transfer information from any node in a ring to any other node with fidelity approaching unity by simply applying bias fields to different nodes. For transfer between nodes 1 and 8, 2 and 7, 3 and 6, or 4 and 5, it suffices to apply a large bias to node 9. But suppose we wish to transfer information from node 1 to 4. Due to translation invariance of the ring we can shift the labels by 2, so that node 1 becomes 3 and 4 becomes 6. Then applying a bias to the new node 9 will enable the transfer.

Further reflection shows that we can achieve maximum transfer fidelities approaching unity for transfer between any pair of nodes in a ring of size \( N \) provided \( N \) is odd by simply biasing the node in the middle between the pair of spins. This is because in this case \( N - 2 \) is odd,
so there must be an odd number of spins along one path around the ring and an even number between the spins around the other. By applying the bias in the middle of the path with an odd number of spins we asymptotically reach a chain with $N - 1$ (even) spins. In this chain the transfer probability between spins mirrored at the centre is 1, which is specifically true for the source and target spin with an even number of spins between them in the chain.

If $N$ is even instead, then the situation is more complicated. If there is an odd number of spins between source and target along the ring, then applying a bias at the middle creates an odd chain where source and target are connected with probability 1 as they are at mirror-symmetric positions in the ring. If there is an even number of spins between source and target, then applying a single bias cannot achieve perfect information transfer as the spins can never be at mirrored positions in the odd chain (which are the only ones in the chain perfectly connected). There are, however, multiple solutions to apply a bias at two spins that can asymptotically generate a suitable chain.

In practice we may be able to simultaneously apply biases to several nodes instead of just a single node. This case is more difficult to treat analytically but preliminary results suggest that numerical optimization can be used in this case to further optimize the applied biases to achieve faster transfers while reducing the magnitude of the required bias fields.
VI. Conclusion

The concept of maximum transfer fidelity for information transfer between nodes in a network of interacting spins was introduced. Criteria for attainability of the bounds in terms of the transition frequencies of the network were given. Attainability was shown to be related, theoretically, to minimality of a linear flow and, computationally, to a translation on a torus. This last connection enabled us to derive upper bounds on the time required to realize transfer fidelities within $\epsilon_{\text{prob}}$ of the maximum transfer fidelity, for arbitrary $\epsilon_{\text{prob}} > 0$, via the simultaneous Diophantine approximation. Algorithms were discussed to find the required approximations.

The general results were applied specifically to regular spin structures such as rings with uniform coupling. In this case, the information transfer infidelity parametric induced by maximum transfer fidelity takes on full significance as it can be shown to be a proper metric defining an information transfer infidelity geometry for the network, which is significantly different from the physical network geometry.

Finally, it was shown how simple controls can be used to change the information transfer fidelities and information transfer geometry of the network. It was shown how this idea can be applied to enable or disable information transfer between a pair of nodes in the network. Simple bias controls are sufficient to direct information flow between nodes. By changing the biases different transfers can be targeted, showing how a spin ring with fixed couplings can be turned into a simple quantum router for information encoded in excitations of a spintronic network.

Directions for future work include optimizing information transfer in spin networks via optimal control to achieve faster and more efficient dynamic routing in more complex spintronic networks. While this work focused on transfer of a single excitation, the concepts and analysis can also be applied to the case of encoding and simultaneous transfer of multiple excitations. This is interesting as it could increase the information transmission capacity of the network. Finally, the effect of noise and decoherence, and protection against the deleterious effect of the environment, should be investigated for specific physical realizations of spintronic networks.

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