Scaled Gromov hyperbolic graphs

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Abstract

In this paper, the δ-hyperbolic concept, originally developed for infinite graphs, is adapted to very large but finite graphs. Such graphs can indeed exhibit properties typical of negatively curved spaces, yet the traditional δ-hyperbolic concept, which requires existence of an upper bound on the fatness δ of the geodesic triangles, is unable to capture those properties, as any finite graph has finite δ. Here the idea is to scale δ relative to the diameter of the geodesic triangles and use the Cartan-Alexandrov-Toponogov (CAT) theory to derive the thresholding value of δ/diam below which the geometry has negative curvature properties.

1 Introduction

A graph property, fundamentally important both in terms of its coarse geometry significance [23] and its potential applicability to networks [15, 10, 13], is probably most easily formulated as the graph looking like a tree when viewed from a distance. A tree is defined as a connected graph such that, given any three vertices, the geodesic triangle made up with the minimum length paths joining them is star shaped. To provide a formal definition of an infinite graph that is nearly a tree, define the fatness $\delta(\triangle)$ of a geodesic triangle $\triangle$ as the minimum of the perimeter of all triangles inscribed to $\triangle$. Should the graph be a tree,

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clearly, $\delta(\triangle) = 0$ for all triangles $\triangle$ embedded in the graph. Then a graph G (or a geodesic metric space X) is said to be δ-hyperbolic if there exists a bound δ_{max} such that $\delta(\triangle) < \delta_{max}$ no matter how distant the vertices of $\triangle$ are. The least such bound is $\delta(G)$ ($\delta(X)$). Such spaces, even though they need not have manifold structure, nevertheless exhibit properties similar to those of simply connected complete Riemannian manifolds M of curvature uniformly bounded from above by $\kappa_{max} < 0$.

Some finite graphs (like trees) unmistakenly exhibit negative curvature properties, other finite graphs (like complete graphs) rather exhibit positive curvature properties, a third class of finite graphs (e.g., those obtained by gluing a tree on a complete graph) have mixed curvature properties, yet the δ-hyperbolic concept is unable to discriminate between the three, as any finite graph G no matter how awesome its size has finite δ. The purpose of this paper is to show that, for very large but finite diameter (geodesic) metric spaces X, the appropriate definition of nonpositive curvature is that the ratio $\delta(\triangle)/\text{diam}(\triangle)$ be less than or equal to $3/2$ for all geodesic triangles $\triangle$ embedded in X (Theorems 2 and 3). The threshold value is derived from a comparison argument with Euclidean space. Such spaces are referred to as *scaled δ-hyperbolic*.

The preceding concept can be transcribed, with some extra caution, to positively curved spaces. Indeed, it can be shown that, for Riemannian manifolds of curvature bounded from below by $\kappa_{min} > 0$, we have $\sup \delta(\triangle)/\text{diam}(\triangle) > 3/2$, at an appropriate scale of the triangle $\triangle$ (see Section 3.2.1).

A variant of the above, which consists in scaling δ relative to the perimeter rather than the diameter, is also considered and is shown to provide an alternative formulation of essentially the same concept. The motivation for the latter is that it is closely related to the more traditional concept of Busemann nonnegatively curved space [16].

The proposed concept allows for some flexibility, as the $\delta(\triangle)/\text{diam}(\triangle) \leq 3/2$ condition could be enforced only for triangles of a diameter bounded from below by some scale R. While the large scale concept would be in the spirit of coarse geometry, the small scale version is related to nonpositive local combinatorial curvature [9] (see, e.g., Theorem 8).

An outline of the paper follows. Section 2 reviews the background material necessary for the understanding of the paper. Section 3 introduces the new concept of scaled δ-hyperbolic spaces. Section 4 develops the local structure compatible with the scaled δ-hyperbolic property. Section 5 shows the relevance of the concept to scale free, heavy tailed graphs [1]. Finally, Section 6 is the conclusion.

2 Background

The context here is that of a *metric space* (X, d). A *path* in X is a continuous mapping $p : [a, b] \to X$. The *length* of a path is defined as $\ell(p) = \sup_{a=a_0<a_1<\ldots<a_{n-1}<a_n=b} \sum_{i=0}^{n-1} d(a_i, a_{i+1})$. A *geodesic* is path $p : [a, b] \to X$ such that $d(p(s), p(s')) = |s - s'|$, $\forall s, s' \in [a, b]$. The metric space X is *geodesic*

if any two points x, y can be joined by a geodesic of length $d(x, y)$.

If v^a, v^b, v^c are three points in a geodesic metric space, a *geodesic triangle* $\triangle v^a v^b v^c$ is defined as $[v^a v^b] \cup [v^b v^c] \cup [v^c v^a]$, where $[v^a v^b]$ denotes a geodesic joining v^a to v^b.

$\mathbb{M}^2_\kappa$ denotes the standard 2-dimensional Riemannian manifold of constant curvature κ. For $\kappa < 0$, it is a hyperboloid; for $\kappa = 0$, it is the Euclidean plane $\mathbb{E}^2$; for $\kappa > 0$, it is the sphere of radius $1/\sqrt{\kappa}$. By the Hopf-Rinow theorem [17, Th. 1.4.8], $\mathbb{M}^2_\kappa$ is a geodesic metric space.

To metrize a graph $G = (V, E)$, where V is the vertex set and E is the edge set, it is convenient to introduce a *length function* $\ell : E(G) \to \mathbb{R}_{>0}$. By topologizing each edge as the unit interval $[0, 1]$ (see [21, Sec. 4.1]), the length function is easily affinely extended to all paths p in G. Then the graph becomes a *length space*. Next, the distance between two points x, y of a graph is defined as the infimum of the length of all paths joining x to y. As such, the graph becomes a metric space. A locally finite graph is geodesic.

Motivated by networking applications where most of the communication cost between vertices x, y is incurred by the routing decision at the vertices traversed by the message, the edge lengths will sometimes be normalized to 1. In those applications, like in Cayley graphs, the distance is relevant only between vertices.

The topological inclusion relation $\triangle \subseteq G$ does not preclude the vertices of $\triangle$ to be in the interior of the edges. We use the notation $\triangle \preceq G$ to indicate that $\triangle$ is a subgraph of G, in which case the vertices of $\triangle$ are vertices of G.

2.1 Comparison theory

The gist of comparison theory is that the metric properties of two triangles isometrically drawn in $\mathbb{M}^2_\kappa$ and $\mathbb{M}^2_{\kappa'}$ are strongly dependent on whether $\kappa \leq \kappa'$ or $\kappa \geq \kappa'$.

Definition 1 ([6, Def. 4.1.8], [5, Chap. II.1],[3, p. 19]) *Given a geodesic triangle $\triangle v^a v^b v^c$ in some geodesic metric space (X, d), the comparison triangle $\bar{\triangle} \bar{v^a} \bar{v^b} \bar{v^c}$ in the standard constant curvature manifold $(\mathbb{M}^2_\kappa, \bar{d})$ is a triangle such that*

$$\begin{aligned}
\bar{d}(\bar{v^a}, \bar{v^b}) &= d(v^a, v^b) \\
\bar{d}(\bar{v^b}, \bar{v^c}) &= d(v^b, v^c) \\
\bar{d}(\bar{v^c}, \bar{v^a}) &= d(v^c, v^a)
\end{aligned}$$

In general, an overbar $(\bar{\cdot})$ is a generic notation to denote a point, a triangle, or the distance in the comparison space.

Proposition 1 ([5, Lemma 1.2.14]) *With the same notation and terminology as above, for $\kappa \leq 0$, the comparison triangle always exists and, for $\kappa > 0$, the comparison triangle exists iff* $\mathrm{perim}(\triangle v^a v^b v^c) \leq \frac{2\pi}{\sqrt{\kappa}}$, *where* perim *denotes the perimeter.*

Definition 2 *With the same notation and terminology as the above, given $x \in [v^b v^c]$, its comparison point in $\bar{\triangle}\bar{v^a}\bar{v^b}\bar{v^c}$ is the point $\bar{x} \in [\bar{v^b}\bar{v^c}]$ such that $d(v^b, x) = \bar{d}(\bar{v^b}, \bar{x})$.*

The comparison triangle also allows a concept of angle to be defined solely in terms of the distance, independently of the concept of inner product.

Definition 3 ([5, Def. 1.12], [6, 4.3]) *The Alexandrov angle $\angle v^c v^a v^b$ at the vertex v^a of a geodesic triangle $\triangle v^a v^b v^c$ is defined as*

$$\angle v^c v^a v^b = \limsup_{\bar{y},\bar{z} \to \bar{v^a}} \angle \bar{y}\bar{v^a}\bar{z}$$

where $\angle \bar{y}\bar{v^a}\bar{z}$ denotes the usual angle at the vertex $\bar{v^a}$ in the comparison triangle $\triangle \bar{v^a}\bar{v^b}\bar{v^c}$ in $\mathbb{M}^2_\kappa$, and $\bar{y} \in [\bar{v^c}\bar{v^a}]$, $\bar{z} \in [\bar{v^a}\bar{v^b}]$ are the comparison points of $y \in [v^c v^a]$, and $z \in [v^a v^b]$, respectively.

The angle $\angle \bar{y}\bar{v^a}\bar{z}$ depends on the metric of $\mathbb{M}^2_\kappa$, but the limit does not depend on what comparison space is chosen [5, Prop. 2.9], [20].

2.2 CAT spaces

Definition 4 ([18, Sec. 3.2],[3, p. 19], [19, Th. VIII.4.1],[6, Sec. 4.1.4]) *The metric space (X, d) is said to be a Cartan-Alexandrov-Toponogov or $\mathrm{CAT}(\kappa \leq 0)$-space if for every geodesic triangle $\triangle v^a v^b v^c$, every point $z \in [v^a v^b]$, and every point $y \in [v^a v^c]$ along with their comparison points in $\mathbb{M}^2_{\kappa \leq 0}$, we have*

$$d(z, y) \leq \bar{d}(\bar{z}, \bar{y})$$

The above is called $\mathrm{CAT}(\kappa \leq 0)$-inequality.

It is easy to see that, for $\kappa' \leq \kappa$, every $\mathrm{CAT}(\kappa')$ space is also a $\mathrm{CAT}(\kappa)$ space.

The concept of $\mathrm{CAT}(\kappa \leq 0)$ space should be obvious: A $\mathrm{CAT}(\kappa \leq 0)$ space is characterized by geodesic triangles that look thinner than those redrawn isometrically in a Riemannian manifold of curvature $\kappa \leq 0$. Since thin triangles are symptomatic of negatively curved spaces, it can be said that a graph has nonpositive curvature if it is a $\mathrm{CAT}(\kappa \leq 0)$ space.

The problem with this nonpositive curvature definition is that *all* triangles at *all scales* must satisfy the CAT(0)-inequality, and this is unlikely to happen in networks, which are too heterogeneous and in which the hyperbolic property occurs only at a large scale. This motivates the new concept introduced in this paper.

2.3 Large scale δ-hyperbolic spaces

We now proceed to a large scale curvature concept for possibly infinite diameter graphs. Because Riemannian manifolds of positive sectional curvature κ

uniformly bounded from below as $\kappa(m) \geq \kappa_{min} > 0$, $\forall m \in M$, have bounded diameter, and because graphs with positive local combinatorial curvature are finite [9, 24, 7], it follows that, at large scale, only nonpositive curvature is relevant. To define such a concept, let (X, d) be a geodesic metric space. The *fatness* of a geodesic triangle $\triangle v^a v^b v^c$ is defined as

$$\delta(\triangle v^a v^b v^c) := \inf \left\{ d(x,y) + d(y,z) + d(z,x) : \begin{array}{r} x \in [v^b v^c] \\ y \in [v^a v^c] \\ z \in [v^a v^b] \end{array} \right\} \quad (1)$$

Next, define

$$\delta(X) := \sup\{\delta(\triangle v^a v^b v^c) : \triangle v^a v^b v^c \subseteq X\} \quad (2)$$

Definition 5 *A geodesic metric space X is said to be (Gromov) δ-hyperbolic if $\delta := \delta(X) < \infty$.*

To understand the spirit of this definition, it is instructive to go back to Riemannian geometry.

Proposition 2 ([23, pp. 84-85]) *Let M be a simply connected complete Riemannian manifold of sectional curvature bounded from above as $\kappa(m) \leq \kappa_{max} < 0$, $\forall m \in M$. Then*

$$\delta(M) \leq \frac{6}{\sqrt{-\kappa_{max}}}$$

In a certain sense, a metric space is δ-hyperbolic if it behaves metrically in the large scale as a negatively curved Riemannian manifold.

Since a locally finite graph is a geodesic metric space, we define such a graph G to be *δ-hyperbolic* if $\delta := \delta(G) < \infty$. It will be shown in Theorem 1 that, for $\triangle \preceq G$, the infimum in (1) occurs on the vertices of the sides of $\triangle$.

3 New concept: scaled δ-hyperbolic space

Here, instead of the CAT(0) approach outlined in Section 2.2, we develop a Gromov-like analysis. Since for any finite graph δ is finite, the Gromov concept becomes meaningful only after scaling $\delta(\triangle)$ by the diameter (or the perimeter) of the triangle $\triangle$.

The *diameter* of a triangle, or any subset of a geodesic metric space for that matter, is defined as $\text{diam}(\triangle) := \sup_{x,y \in \triangle} d(x,y)$. In $\mathbb{M}^2_{\kappa \leq 0}$, $\text{diam}(\triangle v^a v^b v^c) = \max\{d(v^a,v^b), d(v^b,v^c), d(v^c,v^a)\}$. Indeed, for each $\kappa \leq 0$, the manifold $\mathbb{M}^2_\kappa$ is a CAT(0) space [16, Corollary 2.1.3] and therefore the above diameter property follows from [16, 2.3.1]. For $\triangle \preceq G$, it is not in general true that the diameter is achieved on the vertices and this motivates the *vertex diameter*, $\text{vdiam}(\triangle v^a v^b v^c) = \max\{d(v^a,v^b), d(v^b,v^c), d(v^c,v^a)\}$. The perimeter of a triangle $\triangle v^a v^b v^c$ in a geodesic space, on the other hand, does not involve the subtleties of the diameter and is trivially defined as $\text{perim}(\triangle v^a v^b v^c) = d(v^a,v^b) + d(v^b,v^c) + d(v^c,v^a)$.

The proposed concept, $\delta(\triangle)/\text{diam}(\triangle) \leq 3/2$, is related, but not quite equivalent, to the CAT(0) inequality. The CAT(0) inequality implies $\delta(\triangle)/\text{diam}(\triangle) \leq 3/2$ (Corollary 2), the converse holds for a surface (Theorem 4), but as we prove by a counterexample (end of Section 3.2), on a discrete structure, $\delta(\triangle)/\text{diam}(\triangle) \leq 3/2$ does not in general imply CAT(0).

3.1 Combinatorial fatness

Since the new concept emerged out of graph problems, before looking at the scaling of δ, we first investigate the combinatorial aspect of the fatness.

Theorem 1 *For a metric graph $(G = (V, E), d)$, given a geodesic triangle $\triangle v^a v^b v^c$ with its vertices v^a, v^b, v^c in $V(G)$, there exists a solution x, y, z of (1) on the vertices, that is,*

$$\inf \left\{ d(x,y) + d(y,z) + d(z,x) : \begin{array}{c} x \in [v^b v^c] \\ y \in [v^a v^c] \\ z \in [v^a v^b] \end{array} \right\}$$

$$= \inf \left\{ d(x,y) + d(y,z) + d(z,x) : \begin{array}{c} x \in V([v^b v^c]) \\ y \in V([v^a v^c]) \\ z \in V([v^a v^b]) \end{array} \right\}$$

where $V([v^a v^b])$ denotes the vertex set of the side $[v^a v^b]$ of the triangle.

Proof. Let the optimum points x, y, z be in edges $[b_i c_j] \subseteq [v^b v^c]$, $[c_k a_\ell] \subseteq [v^c v^a]$, $[a_m b_n] \subseteq [v^a v^b]$, respectively. Assume first that the edges $[b_i c_j]$, $[c_k a_\ell]$, $[a_m b_n]$ are pairwise nonintersecting. Assume by contradiction that the infimum is reached for x in the open edge $(b_i, c_j) := [b_i c_j] \setminus \{b_i, c_j\}$. Clearly, in this case, $[yx] \ni b_i$ or $[yx] \ni c_j$, since b_i, c_j are the only end vertices from x to y, which by assumption lies in another edge. Then either both geodesics $[yx]$ and $[zx]$ pass through the same end vertex of $[b_i c_j]$ or they pass through different end vertices. The first case where the two geodesics $[yx], [zx]$ pass through the same vertex, say b_i, is impossible, because taking $x = b_i$ would result in a lower length. The remaining possibility is both geodesics passing through different end vertices, say, $[zx] \ni b_i$ and $[yx] \ni c_j$. In the latter case, the length is independent on the position of $x \in [b_i c_j]$. Hence we have the freedom to choose, say, $x = b_i$ so that the optimum length can be achieved for x on a vertex. By a similar argument, the optimum length can be achieved for y, z on vertices as well. Again, a similar argument takes care of the case where any pair of vertices x, y, z are in intersecting edges. ■

3.2 Diameter scaling

Theorem 2 *Fix $\kappa < 0$ and let $R > 0$ be a scale. Then*

$$\sup_{\substack{\triangle \subset \mathbb{M}^2_{\kappa<0} \\ \text{diam}(\triangle) \geq R}} \frac{\delta(\triangle)}{\text{diam}(\triangle)} < \sup_{\triangle \subset \mathbb{E}^2} \frac{\delta(\triangle)}{\text{diam}(\triangle)} \tag{3}$$

Proof. Let $\triangle v^a v^b v^c$ be a triangle in $\mathbb{M}^2_{\kappa<0}$, which, in addition to the diameter constraint, satisfies insize$(\triangle v^a v^b v^c) \geq \epsilon$ for some $\epsilon > 0$. (The *insize* is the diameter of the inscibed triangle [5, Chap. III.H, Def. 1.16].) Define $c := d(v^a, v^b)$, $a := d(v^b, v^c)$, $b := d(v^c, v^a)$. Because of the insize condition, the quantities a, b, c satisfy the strict triangle inequalities, i.e., the triangle $\triangle v^a v^b v^c$ is not flat [5, Def. III.H.1.16]. Let $\triangle \bar{v^a}\bar{v^b}\bar{v^c}$ be a comparison triangle in $\mathbb{E}^2$. Take $x \in [v^b v^c]$, $y \in [v^a v^c]$, $z \in [v^a v^b]$ and let $\bar{x} \in [\bar{v^b}\bar{v^c}]$, $\bar{y} \in [\bar{v^a}\bar{v^c}]$, $\bar{z} \in [\bar{v^a}\bar{v^b}]$ be the corresponding comparison points in $\mathbb{E}^2$. Since $\mathbb{M}^2_\kappa$ is a CAT(κ) space, and hence a CAT(0) space (see [5, Th. II.1.12]), the CAT(0) inequality yields

$$\begin{aligned} d(x,y) &\leq d(\bar{x},\bar{y}) \\ d(y,z) &\leq d(\bar{y},\bar{z}) \\ d(z,y) &\leq d(\bar{z},\bar{y}) \end{aligned}$$

We now prove that the inequalities can be strengthened to strict inequalities, e.g., $d(x,y) < d(\bar{x},\bar{y})$. Let $\triangle \tilde{v^c}\tilde{x}\tilde{y} \subset \mathbb{M}^2_\kappa$ be a comparison triangle of $\triangle \bar{v^c}\bar{x}\bar{y} \subset \mathbb{E}^2$. Draw the geodesic rays $\tilde{v^c}/\tilde{x}$ and $\tilde{v^c}/\tilde{y}$ and pick points $\tilde{v^b} \in \tilde{v^c}/\tilde{x}$ and $\tilde{v^a} \in \tilde{v^c}/\tilde{y}$ such that $d(\tilde{v^c}, \tilde{v^b}) = a$ and $d(\tilde{v^c}, \tilde{v^a}) = b$. Let $\tilde{c} := d(\tilde{v^b}, \tilde{v^a})$. To disprove $d(x,y) = d(\bar{x},\bar{y})$, it suffices to show that $\tilde{c} < c$. To prove the latter, it suffices to show that $\angle\tilde{v^c} < \angle\bar{v^c}$. Since $\angle\tilde{v^c} \leq \angle\bar{v^c}$ by the Rauch-Toponogov comparison theory [17, Sec. 4.5, Perspectives], it suffices to disprove $\angle\tilde{v^c} = \angle\bar{v^c}$. But the latter equality, along with the hyperbolic law of cosines in $\triangle\tilde{x}\tilde{v^c}\tilde{y}, \triangle v^b v^c v^a \subset \mathbb{M}^2_\kappa$, yields $c = |a \pm b|$ (see Appendix), which contradicts the hypothesis that the triangle cannot be flat. Hence,

$$d(x,y) + d(y,z) + d(z,y) < d(\bar{x},\bar{y}) + d(\bar{y},\bar{z}) + d(\bar{z},\bar{y})$$

Taking the minimum of the right-hand side yields

$$\begin{aligned} & d(\hat{x},\hat{y}) + d(\hat{y},\hat{z}) + d(\hat{z},\hat{y}) \\ & < \min\left\{ d(\bar{x},\bar{y}) + d(\bar{y},\bar{z}) + d(\bar{z},\bar{y}) : \begin{array}{c} \bar{x} \in [\bar{v^b}\bar{v^c}] \\ \bar{y} \in [\bar{v^a}\bar{v^c}] \\ \bar{z} \in [\bar{v^a}\bar{v^b}] \end{array} \right\} = \delta(\triangle\bar{v^a}\bar{v^b}\bar{v^c}) \end{aligned}$$

where $\hat{x} \in [v^b v^c]$, $\hat{y} \in [v^a v^c]$, $\hat{z} \in [v^a v^b]$ are the comparison points in $\mathbb{M}^2_\kappa$ of the optimal $\bar{x}, \bar{y}, \bar{z}$ points in $\mathbb{E}^2$. Taking the minimum of the left-hand side of the preceding inequality yields,

$$\min\left\{ d(x,y) + d(y,z) + d(z,y) : \begin{array}{c} x \in [v^b v^c] \\ y \in [v^a v^c] \\ z \in [v^a v^b] \end{array} \right\} < \delta(\triangle\bar{v^a}\bar{v^b}\bar{v^c})$$

Hence
$$\delta(\triangle v^a v^b v^c) < \delta(\triangle\bar{v^a}\bar{v^c}\bar{v^c})$$

Recall that, in a nonpositively curved CAT(0) space, the diameter of a triangle is the largest side length. By construction, the two triangles have the same side lengths, so that

$$\frac{\delta(\triangle v^a v^b v^c)}{\mathrm{diam}(\triangle v^a v^b v^c)} < \frac{\delta(\triangle \bar{v^a} \bar{v^c} \bar{v^c})}{\mathrm{diam}(\triangle \bar{v^a} \bar{v^c} \bar{v^c})}$$

Next, we take the supremum of the right-hand side and show that the strict inequality prevails. Since $\delta(\triangle)/\mathrm{diam}(\triangle) \leq (6/\sqrt{-\kappa})/\mathrm{diam}(\triangle)$, the supremum can be sought over triangles of bounded diameter, say, $\mathrm{diam}(\triangle) \leq R'$. The supremum can also be sought without the insize contraint, since $\mathrm{insize}(\triangle) = 0$ would imply $\delta(\triangle) = 0$. Hence, the supremum can be sought over the compact set $\{(a, b, c) \in \mathbb{R}^3_+ : a \geq b, a \geq c, a \leq R', a \leq b + c\}$. Consequently, after taking the supremum, the strict inequality still holds:

$$\sup_{\substack{\triangle v^a v^b v^c \subset \mathbb{M}^2_\kappa \\ \mathrm{diam}(\triangle v^a v^v v^c) \geq R}} \frac{\delta(\triangle v^a v^c v^c)}{\mathrm{diam}(\triangle v^a v^c v^c)} < \frac{\delta(\triangle \bar{\hat{v^a}} \bar{\hat{v^b}} \bar{\hat{v^c}})}{\mathrm{diam}(\triangle \bar{\hat{v^a}} \bar{\hat{v^b}} \bar{\hat{v^c}})}$$

where $\bar{\hat{v^a}}, \bar{\hat{v^b}}, \bar{\hat{v^c}}$ are the comparison points in $\mathbb{E}^2$ of the optimum points $\hat{v^a}, \hat{v^b}, \hat{v^c}$ achieving the left hand side supremum. Finally, taking the supremum of the right-hand side yields

$$\sup_{\substack{\triangle v^a v^b v^c \subset \mathbb{M}^2_\kappa \\ \mathrm{diam}(\triangle v^a v^b v^c) \geq R}} \frac{\delta(\triangle v^a v^b v^c)}{\mathrm{diam}(\triangle v^a v^b v^c)} < \sup_{\substack{\triangle \bar{v^a} \bar{v^b} \bar{v^c} \subset \mathbb{E}^2 \\ \mathrm{diam}(\triangle v^a v^b v^c) \geq R}} \frac{\delta(\triangle \bar{v^a} \bar{v^b} \bar{v^c})}{\mathrm{diam}(\triangle \bar{v^a} \bar{v^c} \bar{v^c})}$$

Finally, observing that the right-hand side fraction is scale-independent, the restriction on the diameter can be dropped and the result follows. ∎

Theorem 3

$$\sup_{\triangle \subset \mathbb{E}^2} \frac{\delta(\triangle)}{\mathrm{diam}(\triangle)} = \frac{3}{2}$$

for the Euclidean space endowed with the usual metric.

Proof. Consider a Euclidean triangle $\triangle v^a v^b v^c$. Take $x \in [v^b v^c]$, $y \in [v^a v^c]$, and $z \in [v^a v^b]$. The incidence angle at x is defined as $\angle v^b x z$ and the reflection angle at x is defined as $\angle v^c x y$. Recall that, by the Fermat principle, at optimality, that is, when the fatness is achieved, the incidence angle equals the reflection angle. Set $\iota_x = \angle v^b x z = \angle v^c x y$, with a similar definition at y, z. Let α, β, γ be the angles at the vertices v^a, v^b, v^c, respectively. It is readily seen that

$$\begin{aligned} \beta + \iota_x + \iota_z &= \pi \\ \gamma + \iota_x + \iota_y &= \pi \\ \alpha + \iota_z + \iota_y &= \pi \end{aligned}$$

and solving the above for $\iota_x, \iota_y, \iota_z$ yields

$$\iota_x = \alpha, \quad \iota_y = \beta, \quad \iota_z = \gamma$$

It follows that $\triangle v^a zy$ is similar to $\triangle v^a v^b v^c$, with similar statements for $\triangle v^b zx$ and $\triangle v^c xy$. This yields,

$$\begin{array}{ccccc} \frac{|v^a z|}{b} & = & \frac{b-|v^c y|}{c} & = & \frac{|yz|}{a} \\ \frac{|v^c y|}{a} & = & \frac{a-|v^b x|}{b} & = & \frac{|yx|}{c} \\ \frac{|v^b x|}{c} & = & \frac{c-|v^a z|}{a} & = & \frac{|zx|}{b} \end{array}$$

where $|\cdot|$ denotes the usual length of a Euclidean line segment. Solving the above for $|v^b x|, |v^c y|, |v^a z|$ and plugging the solution in

$$\delta = |xy| + |yz| + |zx| = \frac{a|v^a z|}{b} + \frac{c|v^c y|}{a} + \frac{b|v^b x|}{c}$$

yields

$$\delta(\triangle v^a v^b v^c) = \frac{2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4}{2abc}$$

To prove that $\delta/\text{diam} \leq \frac{3}{2}$, take, without loss of generality, $a, b \leq c = 1$, in which case, it suffices to show that

$$f(a,b) \equiv a^4 + b^4 + 1 - 2a^2b^2 - 2a^2 - 2b^2 + 3ab \geq 0$$

$\forall (a,b) \in [0,1]^2$ satisfying the triangle inequality $1 \leq a + b$. Nominally, such a problem would be tackled as a Tarski-Seidenberg decision problem, but the latter does not, to our knowledge, exploit the symmetric property of the polynomial, viz., $f(a,b) = f(b,a)$. To exploit this symmetry, recall that any smooth symmetric function can be written as a smooth function of the elementary symmetric functions [2, Sec. 10.7], which in this case reduce to $\sigma_1 = a + b$ and $\sigma_2 = ab$. The triangle inequality reads $\sigma_1 \geq 1$. Next to this, we have $\sigma_2 \leq 1$ and $\sigma_1 \leq 2$. Also we need $\sigma_1^2 \geq 4\sigma_2$ to secure real a, b. Applying this basic fact about symmetric functions, it is easily observed that

$$f(a,b) = \sigma_2(-4\sigma_1^2 + 7) + (\sigma_1^2 - 1)^2$$

and we have to show that the above function is nonnegative for all σ's satisfying the constraints. The problem is that $-4\sigma_1^2 + 7 < 0$ for $\sqrt{\frac{7}{4}} < \sigma_1 < 2$. Therefore the minimum of f is attained by taking σ_2 as large as possible, that is, $\sigma_2 = \frac{\sigma_1^2}{4}$ so that it suffices to verify that

$$\frac{\sigma_1^2}{4}(-4\sigma_1^2 + 7) + (\sigma_1^2 - 1)^2 \geq 0$$

for all $\sigma_1 \in (1,2)$. But the above is equivalent to

$$-\sigma_1^2 + 4 \geq 0$$

and the inequality is obvious under the constraint $\sigma_1 < 2$.

■

The preceding theorem holds true in $\mathbb{R}^2$ endowed with an arbitrary positive definite metric. Indeed, such a Riemannian metric on $\mathbb{R}^2$ is flat, hence has the same geodesics as in Theorem 3, and everything follows.

Observe that failure to enforce a lower bound on the scale R in Eq. (3) the strict inequality would not hold. The reason is that the supremum would be reached for an arbitrarily small triangle. Indeed, an infinitesimally small triangle can be thought to be in the tangent space, in which Euclidean geometry prevails, hence making $\sup \delta(\triangle)/\mathrm{diam}(\triangle) = 3/2$. Hence,

Corollary 1 *$\sup_{\triangle \subset \mathbb{M}^2_\kappa} \delta(\triangle)/\mathrm{diam}(\triangle)$ is constant for $\kappa < 0$ and left-continuous at $\kappa = 0$.*

It is observed that this continuity no longer holds for the δ of the Gromov 4-point condition [12].

With the above, we can define the new concept:

Definition 6 *A metric space (X, d) is said to be (diameter) scaled Gromov hyperbolic if*

$$\sup_{\substack{\triangle \subseteq X \\ \mathrm{diam}(\triangle) \geq R > 0}} \frac{\delta(\triangle)}{\mathrm{vdiam}(\triangle)} < \frac{3}{2}.$$

We now investigate the extent to which this new definition is related to the more traditional nonpositive curvature concepts.

Corollary 2 *Let X be a $\mathrm{CAT}(0)$ (resp., $\mathrm{CAT}(\kappa < 0)$) metric space. Then, for any scale $R > 0$,*

$$\sup_{\triangle \subseteq X} \frac{\delta(\triangle)}{\mathrm{vdiam}(\triangle)} \leq \frac{3}{2} \qquad \left(\textit{resp.,} \quad \sup_{\substack{\Delta \subseteq X \\ \mathrm{diam}(\Delta) \geq R}} \frac{\delta(\triangle)}{\mathrm{vdiam}(\triangle)} < \frac{3}{2} \right)$$

Proof. Let v^a, v^b, v^c be 3 vertices in a CAT(0) space and let $\bar{v^a}, \bar{v^b}, \bar{v^c}$ be the comparison points in $\mathbb{E}^2$. Next, if we pick $x \in [v^b v^c]$, $y \in [v^a v^c]$, $z \in [v^a v^b]$, the CAT(0) inequality yields

$$d(x, y) + d(y, z) + d(z, x) \leq d(\bar{x}, \bar{y}) + d(\bar{y}, \bar{z}) + d(\bar{z}, \bar{x})$$

and an argument similar to that of Theorem 2 yields

$$\inf \left\{ d(x, y) + d(y, z) + d(z, x) : \begin{array}{r} x \in [v^b v^c] \\ y \in [v^a v^c] \\ z \in [v^a v^b] \end{array} \right\}$$
$$\leq \inf \left\{ d(\bar{x}, \bar{y}) + d(\bar{y}, \bar{z}) + d(\bar{z}, \bar{x}) : \begin{array}{r} \bar{x} \in [\bar{v^b} \bar{v^c}] \\ \bar{y} \in [\bar{v^a} \bar{v^c}] \\ \bar{z} \in [\bar{v^a} \bar{v^b}] \end{array} \right\}$$

Using $\text{vdiam}(\triangle v^a v^b v^c) = \text{vdiam}(\triangle \bar{v^a}\bar{v^b}\bar{v^c})$ and Theorem 3, it follows that

$$\frac{\delta(\triangle)}{\text{vdiam}(\triangle)} \leq \frac{\delta(\bar{\triangle})}{\text{vdiam}(\bar{\triangle})} \leq \frac{3}{2}$$

and the corollary is proved for $\kappa = 0$. The proof for $\kappa < 0$ is the same, except that we use $\mathbb{M}^2_\kappa$ as comparison space and invoke Theorems 2 in addition to Theorem 3. ■

The converse of the preceding corollary does not hold. Specifically, if a triangle $\triangle$ in a metric space has the property $\delta(\triangle)/\text{vdiam}(\triangle) < 3/2$, it need not satisfy even the CAT(0) inequality. Indeed, consider a metric space made up of a circle of length 18 on which 3 line segments $[xy]$, $[yz]$, $[zx]$, metrized as

$$\ell([xy]) = 4, \quad \ell([yz]) = 0.5, \quad \ell([zx]) = 4,$$

have been attached. In this metric space, construct $\triangle v^a v^b v^c$ such that

$$\begin{array}{ll} d(y, v^a) = 3, & d(v^a, z) = 3 \\ d(z, v^b) = 3, & d(v^b, x) = 3 \\ d(x, v^c) = 3, & d(v^c, y) = 3 \end{array}$$

It is trivial to verify that $\frac{d(x,y)+d(y,z)+d(z,x)}{\text{(v)diam}(\triangle v^a v^b v^c)} < \frac{3}{2}$, so that $\frac{\delta(\triangle v^a v^b v^c)}{\text{(v)diam}(\triangle v^a v^b v^c)} < \frac{3}{2}$ for both diameter concepts. Clearly, in the comparison triangle $\triangle \bar{v^a}\bar{v^b}\bar{v^c} \subset \mathbb{E}^2$, we have $d(\bar{z}, \bar{x}) = 3$. Since $4 = d(z, x) > d(\bar{z}, \bar{x}) = 3$, the CAT(0) inequality does not hold.

Despite the above counterexample, the following is true:

Theorem 4 *If a simply connected complete 2-dimensional manifold S is such that, for some $\epsilon > 0$, we have $\frac{\delta(\triangle)}{\text{diam}(\triangle)} \leq \frac{3}{2}$, $\forall \triangle \subset S$, $\text{diam}(\triangle) < \epsilon$, then $\kappa(s) \leq 0$, $\forall s \in S$, so that S is a CAT(0)-space.*

Proof. Indeed, if there exists a point s such that $\kappa(s) > 0$, there exists a neighborhood ball $B_{r \leq \epsilon/2}(s)$ of s, such that, $\forall q \in B_r(s)$, $0 < \kappa_{min} \leq \kappa(q)$. Construct an equilateral (i.e., $a = b = c$) geodesic triangle $\triangle v^a v^b v^c$ in $B_r(s)$ along with its comparison triangle $\triangle \bar{v^a}\bar{v^b}\bar{v^c}$ in $\mathbb{M}^2_{\kappa_{min}}$. By the Rauch-Toponogov comparison theory (see [17, Sec. 4.5, Perspectives], [6, Sec. 10.3]), it follows that $\angle \bar{v^b}\bar{v^a}\bar{v^c} \leq \angle v^b v^a v^c$. Now, take $z \in [v^a v^b]$ and $y \in [v^a v^c]$ along with their comparison points $\bar{z}$, $\bar{y}$. From [6, Th 4.3.5], it follows that $d(\bar{y}, \bar{z}) \leq d(y, z)$. Hence

$$d(\bar{x}, \bar{y}) + d(\bar{y}, \bar{z}) + d(\bar{z}, \bar{x}) \leq d(x, y) + d(y, z) + d(z, x)$$

where $x \in [v^b v^c]$ and $\bar{x}$ is its comparison point. Using an argument similar to the one of Theorem 2 (but with the reverse inequality since here the curvature is nonnegative) we get

$$\delta(\triangle v^a v^b v^c) \geq \delta(\triangle \bar{v^a}\bar{v^b}\bar{v^c})$$

In the comparison triangle, because of the symmetry of the problem ($\triangle \bar{v^a}\bar{v^b}\bar{v^c}$ is equilateral), it is easily seen that the δ is achieved for $\bar{x}$, $\bar{y}$, $\bar{z}$ at the midpoints. From there on, a little bit of spherical trigonometry yields

$$\frac{\delta(\triangle \bar{v^a}\bar{v^b}\bar{v^c})}{a} > \frac{3}{2}$$

where $a = \ell([v^b v^c])$. It follows that

$$\frac{\delta(\triangle v^a v^b v^c)}{a} > \frac{3}{2}$$

A contradiction. ■

3.2.1 positive curvature

It transpires from the preceding theorem that, in the standard manifold $\mathbb{M}^2_{\kappa>0}$, we have $\sup_{\triangle} \frac{\delta(\triangle)}{\mathrm{diam}(\triangle)} > \frac{3}{2}$, at a scale not too small ($\mathrm{diam}(\Delta) > 0$) and not too large (before the geodesics converge, that is, $\mathrm{diam}(\Delta) < \frac{\pi}{2\sqrt{\kappa}}$).

As an illustration, it is instructive to consider the complete graph K_n on n vertices, all of unit length. First, it is easily observed that the complete graph is positively curved by the intuitive clustering coefficient definition of [8]. From a more precise standpoint, as a consequence of [7, Theorem 1.7], K_n has a 2-cell embedding in either S^2 of $\mathbb{PR}^2$. As far as isometric embedding is concerned, the following can be said (see [15]):

Proposition 3 *The complete graph K_n with unit length edge is isometrically embeddable in the $(n-2)$-sphere of radius $\frac{1}{\cos^{-1}(-\frac{1}{n-1})}$, and $(n-2)$ is the least such dimension.*

Proof. Let $V = \{v^i : i = 1, ..., n\}$ be a vertex set along with a distance function d. A general result [4, Chap. VII] says that the metric space (V, d) is isometrically embeddable in a sphere of least dimension iff the $n\times n$ Gram matrix $\{\cos(d(v^i, v^j)\sqrt{\kappa})\}_{1\le i,j\le n}$ is positive semidefinite of least rank for some κ such that $\mathrm{diam}(V) \le \pi/\sqrt{\kappa}$. In this particular case, the Gram matrix is Toeplitz, with 1's on the diagonal, and $c := \cos(\sqrt{\kappa}) < 1$ off the diagonal. A little bit of analysis shows that the $k \times k$ principal minor is $(1-c)^{k-1}((k-1)c+1)$. It is also easily verified that the sequence of nested principal minors decreases with k. Hence to secure minimum dimension embedding $(n-1)c+1 = 0$, from which the result follows. ■

It turns out that K_n is also positively curved by our definition: indeed, it is readily verified that $\forall \triangle \preceq K_n$, $\frac{\delta(\triangle)}{\mathrm{vdiam}(\triangle)} = 2 > 3/2$.

3.3 Perimeter scaling

Theorem 5 *Fix $\kappa < 0$ and let $R > 0$ be a scale. Then*

$$\sup_{\substack{\triangle \subset \mathbb{M}^2_{\kappa<0} \\ \mathrm{diam}(\triangle) \ge R}} \frac{\delta(\triangle)}{\mathrm{perim}(\triangle)} < \sup_{\triangle \subset \mathbb{E}^2} \frac{\delta(\triangle)}{\mathrm{perim}(\triangle)}$$

Proof. Same as that of Theorem 2. ■

Theorem 6

$$\sup_{\triangle \subset \mathbb{E}^2} \frac{\delta(\triangle)}{\text{perim}(\triangle)} = \frac{1}{2}$$

for a Euclidean space endowed with the usual metric.

Proof. Consider a Euclidean triangle $\triangle v^a v^b v^c$. Using the result of the diameter-scaled case, we get

$$\frac{\delta(\triangle v^a v^b v^c)}{\text{perim}(\triangle v^a v^b v^c)} = \frac{2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4}{2abc(a+b+c)}$$

To prove that $\delta/\text{perim} \leq \frac{1}{2}$, take, without loss of generality, $a, b \leq c = 1$, in which case, it suffices to show that

$$f(a,b) \equiv a^4 + b^4 + 1 - 2a^2b^2 - 2a^2 - 2b^2 + (a+b+1)ab \geq 0$$

$\forall (a,b) \in [0,1]^2$ satisfying the triangle inequality $1 \leq a + b$. As in the proof of Theorem 3, let $\sigma_1 = a + b$ and $\sigma_2 = ab$. The triangle inequality reads $\sigma_1 \geq 1$. Next to this, we have $\sigma_2 \leq 1$ and $\sigma_1 < 2$. Also we need $\sigma_1^2 \geq 4\sigma_2$ to secure real a, b. Applying this basic fact about symmetric functions, it is easily observed that

$$f(a,b) = \sigma_2(-4\sigma_1^2 + \sigma_1 + 5) + (\sigma_1^2 - 1)^2$$

and we have to show that the above is nonnegative for all σ_1 and σ_2 satisfying the constraints. The problem is that $-4\sigma_1^2 + \sigma_1 + 5 < 0$ for $\frac{5}{4} < \sigma_1 < 2$. Therefore the minimum of f is attained by taking σ_2 as large as possible, that is, $\sigma_2 = \frac{\sigma_1^2}{4}$ so that it suffices to verify that

$$\frac{\sigma_1^2}{4}(-4\sigma_1^2 + \sigma_1 + 5) + (\sigma_1^2 - 1)^2 \geq 0$$

for all $\sigma_1 \in (1,2)$. But the above is equivalent to

$$(\sigma_1 - 2)^2(\sigma_1 + 1) \geq 0$$

and the inequality is obvious. ■

3.3.1 positive curvature

The complete graph K_n is also positively curved by the perimeter scaling criterion. Indeed, $\forall \triangle \preceq K_n$, $\delta(\triangle)/\text{perim}(\triangle) = 2/3 > 1/2$.

3.4 Connection with Busemann nonnegatively curved spaces

The perimeter scaled case has a close connection with the Busemann concept of nonnegatively curved spaces. Recall that a metric space (X, d) is said to be a *global Busemann nonnegatively curved space* [16, Def. 2.2.4, 2.2.5] if $\forall v^a, v^b, v^c \in X$, $d(m(v^a, v^b), m(v^a, v^c)) \leq \frac{1}{2}d(v^b, v^c)$, where $m(v, w)$ denotes a midpoint of a geodesic $[vw]$. If we strengthen this condition to $d(m(v^a, v^b), m(v^a, v^c)) < \frac{1}{2}d(v^b, v^c)$, but to hold at a scale bounded from below, the latter implies that for any geodesic triangle $\triangle v^a v^b v^c$ of positive diameter,

$$\frac{\text{perim}(\triangle m(v^a, v^b)m(v^b, v^c)m(v^c, v^a))}{\text{perim}(\triangle v^a v^b v^c)} < \frac{1}{2}$$

Since the perimeter scaled inequality is achieved for the triangle inscribed at the midpoints, it surely holds for the minimum perimeter triangle; hence $\delta(\triangle)/\text{perim}(\triangle) < 1/2$. Thus a space enjoying the strengthened version of the Busemann inequality is perimeter-scaled Gromov-hyperbolic, but the converse does not hold. This aspect is further developed in [11].

4 local structure

The new relative δ concept allows for flexibility in its implementation, as it can be enforced at various scales, preferably at large scale as per the traditional coarse geometry paradigm. However, as we show in this section, enforcing the condition $\delta(\triangle)/\text{vdiam}(\triangle) < 3/2$ or its perimeter scaled counterpart all the way down to the mesh reveals some local structure, itself related to a local curvature concept.

To reach this low scale, we recursively decompose a triangle by drawing the minimum perimeter inscribed triangle: $\triangle v^a v^b v^c$ is broken down into $\triangle v^a yz$, $\triangle v^b xz$, $\triangle v^c xy$, $\triangle xyz$; then each such triangle is itself broken down into 4 triangles using the same procedure; etc. The recursion will end up with three branch stars and/or cycles that can no longer be decomposed. (Formally, these cycles c cannot be decomposed by surgering c along an edge-path with its end points on c (see [5, Chap. III.H, Def. 210]).) By *local structure*, we mean the restriction on those indecomposable cycles imposed by the scaled δ-hyperbolic condition. The motivation for looking at those indecomposable cycles is two-fold:

1. As we show in this Section, the indecomposable cycles turn out to be m-gons on *specific* numbers of edges. Consequently, a scaled Gromov hyperbolic graph would consist, locally around a vertex v^a, of m-gons glued along common edges incident upon v^a. As such, this *local structure* allows for a connection with a local graph curvature concept [7, 9, 24], a concept heavily dependent on the number of edges of those faces of the graph incident upon v^a.

2. Those indecomposable cycles provide an area functional, $A(\triangle)$, defined to be the number of cells bounded by the indecomposable cycles. (The

triangle Δ along with the geodesics beginning/ending on its vertices make a graph G_Δ that can be embedded in the compact surface S_g [22, Sec. 3.3]; the *cells* are formally defined as the connected components of $S_g \setminus G_\Delta$.) It is conjectured that, in a scaled δ-hyperbolic space, $A(\triangle)$ is subquadratic, superlinear in $\mathrm{perim}(\triangle)$ [11].

Let $c\{m\}$ denote the cycle graph on m edges. We begin by computing the $\delta(c\{m\})$ as per the traditional Gromov analysis and then we consider the various scalings.

Any triangle $\triangle \preceq c\{m\}$ is uniquely characterized up to isometry by the number of unit length edges in each of its sides, say m_1, m_2, m_3, with $m_1 + m_2 + m_3 = m$ and subject to the triangle inequalities. By the S_3-symmetry of the problem, it can be assumed that $m_1 \leq m_2 \leq m_3$, and the triangle inequalities reduce to $m_3 \leq m_1 + m_2$.

Proposition 4

$$\max_{\triangle \preceq c\{m\}} \delta(\triangle) = 2 \left\lfloor \frac{m}{3} \right\rfloor$$

Proof. Clearly,

$$\max_{\triangle \preceq c\{m\}} \delta(\triangle) = \max_{m_1+m_2+m_3=m} 2 \min\{m_1, m_2, m_3\}$$

Write $m = 3k+l$, where $l = 0, 1, 2$. Consider the subset of points $(m_1, m_2, m_3) \in \mathbb{N}^3$ such that $m_1 + m_2 + m_3 = m$. Taking the equality constraint into consideration, the string of inequalities becomes $m_1 \leq m_2 \leq m - m_1 - m_2$. Write $m_2 = m_1 + \mu$, with $\mu \in \mathbb{N}$, and the string becomes $m_1 \leq m_1 + \mu \leq m - 2m_1 - \mu$. Clearly, $\min\{m_1, m_2, m_3\} = m_1$. In order to maximize m_1, the only constraint is $m_1 + \mu \leq m - 2m_1 - \mu$, that is, $3m_1 \leq (3k + l) - 2\mu$. Hence $\max m_1$ is attained for $\mu = 0$ and $\max m_1 = k$. For this solution, it is easily verified that the triangle inequality reads $k \geq l$, which is obviously satisfied for $k \geq 2$. Hence $\max_{\triangle \preceq c\{m=3k+l\}} \delta(\triangle) = 2k$ for $k \geq 2$. The case $k = 1$ is treated separately; it only involves three regular polygons: the triangle, the square and the pentagon; it is easily seen that for all such polygons $\delta = 2$. Hence the general result holds for $k \geq 1$. ■

Observe that, if the constraint that the triangles have their vertices in $V(G)$ is dropped, we obtain $\delta(c\{m\}) := \sup_{\triangle \subseteq c\{m\}} \delta(\triangle) = 2m/3$, as easily proved.

4.1 Diameter-scaled spaces

Here we compute $\max\left(\delta(\triangle)/\mathrm{vdiam}(\triangle)\right)$ of the cycle graph $c\{m\}$ on m edges of equal length, and we identify what cycles are consistent with the scaled δ-hyperbolic condition.

Proposition 5

$$\max_{\triangle \preceq c\{m\}} \frac{\delta(\triangle)}{\mathrm{vdiam}(\triangle)} = \begin{cases} 2 & \text{for } m = 3k \\ \frac{2k}{k+1} & \text{for } m = 3k+1, 3k+2 \end{cases}$$

Proof. Clearly,

$$\max_{\triangle \preceq c\{m\}} \frac{\delta(\triangle)}{\text{vdiam}(\triangle)} = \max_{m_1+m_2+m_3=m} \frac{2\min\{m_1,m_2,m_3\}}{\max\{m_1,m_2,m_3\}}$$

As in Proposition 4, we set $m_1 \leq m_2 \leq m_3$.

For $m = 3k$, it is claimed that

$$\max_{m_1+m_2+m_3=m} \frac{2\min\{m_1,m_2,m_3\}}{\max\{m_1,m_2,m_3\}} = 2$$

Indeed, it suffices to show that

$$\frac{2m_1}{m_3} \leq 2$$

and that equality can be reached. The $\leq$ inequality is pretty straightforward; moreover, since $m_1+m_2+m_3 = 3k$, equality is reached for $m_1 = m_2 = m_3 = k$. This optimal solution obviously satisfies the triangle inequality.

For $m = 3k + l$, $l \neq 0$, it is claimed that

$$\max_{m_1+m_2+m_3=m} \frac{2\min\{m_1,m_2,m_3\}}{\max\{m_1,m_2,m_3\}} = \frac{2k}{k+1}$$

Indeed, it suffices to show that

$$\frac{2m_1}{m_3} \leq \frac{2k}{k+1}$$

and that equality can be reached. The above is easily seen to be equivalent to $2k(m_3 - m_1) - 2m_1 \geq 0$. Since $m_3 - m_1 \geq 1$, it follows that $2k(m_3 - m_1) - 2m_1 \geq 2(k - m_1)$, so that it suffices to show that $k \geq m_1$. Indeed, clearly, $3m_1 < m = 3k+l$, so that $k > m_1 - \frac{l}{3}$. But since k is integer, $k \geq m_1$. Equality of the above is achieved for $m_1 = k$, $m_3 = k + 1$ (and $m_2 = k + l - 1$). For this latter solution, the triangle inequality reads $k \geq 2 - l$, which is obviously satisfied for $k \geq 2$. As before, the case $k = 1$ is treated separately and it is easily seen that for the triangle, the square and the pentagon the general result holds true. ■

Observe that, if the graph structure of $c\{m\}$ is traded for a purely topological space, $\text{diam}(\triangle) = m/2$, $\forall \triangle \subseteq c\{m\}$, so that $\sup_{\triangle \subseteq c\{m\}} \delta(\triangle)/\text{diam}(\triangle) = (2m/3)/(m/2) = 4/3$.

Let us come back to the recursive procedure of decomposing a triangle. Assume the procedure stops with a triangle with its vertices on a cycle $c\{m\}$. If the $\delta(\triangle)/\text{vdiam}(\triangle) < 3/2$ condition holds for all triangles, it is clear that the procedure cannot stop with a triangle embedded in a cycle on $m = 3k$ vertices, because by Proposition 5 the latter would yield $\sup_{\triangle \preceq c\{m=3k\}} \delta(\triangle)/\text{vdiam}(\triangle) = 2$. Hence $m = 3k + 1$ or $m = 3k + 2$. In the latter case, the scaled δ-hyperbolic condition along with the preceding theorem yields $\frac{2k}{k+1} < \frac{3}{2}$, which yields $k < 3$. Hence we have the following:

Theorem 7 *Let the graph G be such that $\sup_{\triangle \preceq G} \delta(\triangle)/\text{vdiam}(\triangle) < 3/2$. Then for an arbitrary $\triangle v^a v^b v^c$, the procedure of recursively decomposing the triangle stops with stars and/or triangles embedded in cycles on 4, 5, 7, or 8 vertices.*

The combinatorial local curvature of a graph defined in [9, 24, 7], specifically, $k(v) = 1 - \frac{\deg(v)}{2} + \sum_{\phi_k \succeq v} \frac{1}{|\phi_k|}$, where $|\phi_k|$ is the number of edges of the regular face ϕ_k incident upon v, shows some connection with the above result. As an illustration, consider 3 pentagons, $u^1u^2u^3u^4u^5$, $v^1v^2v^3v^4v^5$, $w^1w^2w^3w^4w^5$, where the vertices are ordered counterclockwise. Glue the 3 pentagons along their edges, $[v^1v^2] = [u^1u^5]$, $[u^1u^2] = [w^1w^5]$, $[w^1w^2] = [v^1v^5]$, so that the vertex $v^1 = u^1 = w^1$ is common to the 3 pentagons, and let G denote the resulting graph. The Higuchi local combinatorial curvature is $k(v^1) = 1 - \frac{3}{2} + 3\frac{1}{5} = \frac{1}{10} > 0$. On the other hand, $\frac{\delta(\triangle v^3u^3w^3)}{\text{vdiam}(\triangle v^3u^3w^3)} = \frac{6}{3}$, so that $\max_{\triangle \preceq G} \frac{\delta(\triangle)}{\text{vdiam}(\triangle)} \geq \frac{6}{3} > \frac{3}{2}$, hence violating the scaled Gromov-hyperbolic condition, as expected.

4.2 Perimeter scaled spaces

For the perimeter scaled case, $\sup_{\triangle \preceq c\{m\}} \delta(\triangle)/\text{perim}(\triangle)$ is clearly achieved when, topologically, $\triangle = c\{m\}$, for otherwise the triangle is flat and $\delta(\triangle) = 0$. But all such triangles have the same perimeter m; hence,

$$\begin{aligned} \sup_{\triangle \preceq c\{m\}} \frac{\delta(\triangle)}{\text{perim}(\triangle)} &= \frac{2\left\lfloor \frac{m}{3} \right\rfloor}{m} \\ &= \frac{2\left\lfloor \frac{3k+l}{3} \right\rfloor}{3k+l} \\ &= \frac{2k}{3k+l} \end{aligned}$$

In order to have $\frac{2k}{3k+l} < \frac{1}{2}$, the only possible cycle is $k = 1$ and $l = 2$; hence the pentagon is the only building block in this case. In this perimeter scaled case, it is possible to establish a connection with the local combinatorial curvature, stronger than in the case of the diameter scaled case.

Theorem 8 *Let G be the (plane) graph obtained after gluing d pentagons of unit edge length around a common vertex v and such that every edge incident upon v is common to exactly 2 pentagons. Then, for $d \geq 4$, $k(v) < 0$ and*

$$\sup_{\substack{\triangle \preceq G \\ \triangle \not\ni v}} \frac{\delta(\triangle)}{\text{perim}(\triangle)} < \frac{1}{2}$$

The restriction that $\triangle$ is not allowed to contain v is to make the scaled δ relevant to the local curvature around v. Indeed, a triangle containing v would disregard some of the pentagons incident upon v and would not be representative of the curvature at v.

Proof. Trivially, $k(v) = (10 - 3d)/10$ and the local combinatorial curvature claim holds.

If $\triangle$ does not contain v, then either the triangle is flat and $\delta(\triangle) = 0$ or $\text{perim}(\triangle) = 3d$ and $\delta(\triangle) \leq 6$. In the former case, the result is proved. In the latter case, $\delta(\triangle)/\text{perim}(\triangle) \leq 2/d$. So, only the case $d = 4$ needs to be examined more carefully. In this case, $\text{perim}(\triangle v^a v^b v^c) = 12$ and the triangle is topologically the union of those edges not common to two pentagons. It follows that there exists at least one side of the triangle, say $[v^a v^b]$, of a length at least 4. But it is easily seen that there exist two points such that any geodesic between them must pass through v, a contradiction. ■

5 Relevance

5.1 A large scale computationally simplified concept

Computing $\delta(\triangle)/\text{vdiam}(\triangle)$ for *all* geodesic triangles $\triangle \preceq G$ is very intensive. However, it need not be that way in the broad class of problems where the primary concern is the behavior of the graph at large scale. In those problems where the coarse geometry paradigm prevails, the issue is the limit

$$\lim_{R\uparrow \text{vdiam}(G)} \sup_{\substack{\triangle \preceq G \\ \text{vdiam}(\triangle) \geq R}} \frac{\delta(\triangle)}{\text{vdiam}(\triangle)}$$

The above limiting concept easily leads to a computationally simplified procedure via the inequality:

$$\sup_{\substack{\triangle \preceq G \\ \text{vdiam}(\triangle) = \text{vdiam}(G)}} \frac{\delta(\triangle)}{\text{vdiam}(\triangle)} \leq \frac{\sup_{\triangle \preceq G} \delta(\triangle)}{\text{vdiam}(G)}$$

The right-hand side of the above inequality is much more easily computable than the main concept; furthermore, it provides an upper bound on the large scale limit of the early concept; in particular, should $\sup_{\triangle \preceq G} \delta(\triangle)/\text{vdiam}(G) < 3/2$, then the graph is scaled Gromov hyperbolic.

5.2 Large scale curvature behavior of classical graph models

To show the relevance of the proposed concept, we investigate the computationally simplified scaled δ-hyperbolic property of four graph models. Recall that the Erdös-Rényi random graph $R(n, M)$ is characterized by n vertices and M edges distributed uniformly at random among all possible $\binom{n}{2}$ edges. However, here, in order to avoid disconnected graphs for which $\delta = \infty$, a random tree of order n is first generated and then the remaining $M - (n-1)$ edges are distributed uniformly at random over the remaining $\binom{n}{2} - n + 1$ possible edges. Call this new model $R'(n, M)$. The Small World $W(n, d, \beta)$ model of Watts and Strogatz [25] is constructed from a regular lattice of n vertices of homogeneous degree d, after rewiring every link with probability β to another node chosen uniformly at random over the entire graph. The scale-free or growth/preferential attachment model $B(n_0, M_0, n, m)$ of Barabási et al. [1] is constructed from a core network of n_0 vertices and of size M_0 after recursively adding $n - n_0$ new vertices each carrying m new edges attached to the previous vertices with a probability proportional to the degree, referred to as preferential attachment. A slight variant is the uniform attachment probability $U(n_0, M_0, n, M)$ model.

To draw a fair comparison among all four generators despite their different parameterization, we fix the order of the graphs n and utilize the total number of edges M as crucial connectivity parameter. For the Small World model, the constraining relationship is $M = \lfloor \frac{dn}{2} \rfloor$, where d is the lattice degree; for the scale-free and uniform attachment models, we have $M = M_0 + (n - n_0)m$. To make the topological graphs generated by all four models geodesic spaces, each edge is declared bi-directional with unit length.

Let E_M denote the mathematical expectation relative to the probability measure of the graph generators R', W, B and U of fixed order n and variable size M. As explained in Section 5.1, the objective here is to evaluate

$$E_M \left(\frac{\max_{\triangle v^a v^b v^c \preceq G} \delta(\triangle v^a v^b v^c)}{\mathrm{vdiam}(G)} \right)$$

Monte Carlo simulation results of the above mathematical expectation versus M for the R', W, B and U generators with $n = 50$ are plotted in Fig. 1. The overall shape of the curves can be explained as follows: In all four cases, the start-up graph is a random tree; when vertices and edges are being added, the δ initially increases rapidly because the tree is being "fattened," while the diameter remains roughly constant; then the fatness decreases because the new edges start creating some shortcuts, while the diameter still remains constant; then as M keeps on increasing we reach the most interesting region, half-way between the minimum and the maximum M, because there the geometry is hyperbolic without trivial tree structure; after that, there are too many edges, which has the effect of decreasing the diameter while the fatness remains constant, hence the curve goes up, to approach the value of 2, which as easily seen is exactly $\frac{\delta(K_{50})}{\mathrm{vdiam}(K_{50})}$, where K_{50} denotes the complete graph on 50 vertices.

It is claimed that the region in which the four generators show different results is the one representative of the asymptotic situation $n \to \infty$. Indeed, in this asymptotic situation, we have $m \times \#\text{vertices} = \#\text{edges}$. In the discrepancy region, we have $n = 50$ vertices and $M = 200$ edges. The parameter m is given as solution to the edge counting equation, viz., $200 = 10 - 1 + (50 - 10)m$, which yields $m \approx 5$. With these values, the asymptotic condition $(m = 5) \times 50$ vertices ≈ 200 edges approximately holds.

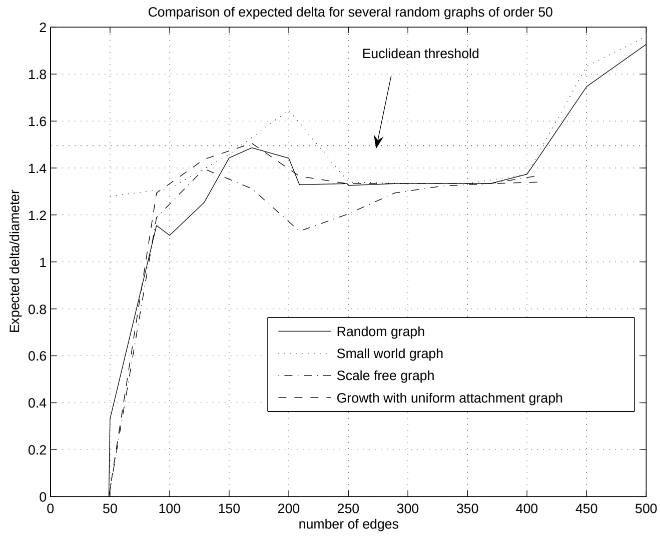

Figure 1: The mathematical expectation of $\max(\delta(\triangle)/\mathrm{diam}(G))$ versus the total number of edges M for all 4 graph generators. Observe that the scale free graph is the most hyperbolic.

Probably the most important observation in the discrepancy region is that the scale-free model is the most hyperbolic while the Small World model is the least hyperbolic. Another observation is that the preferential attachment graphs are more hyperbolic than the uniform attachment graphs, which is not surprising since the preferential attachment generator has heavy-tailed degree distribution as $n \to \infty$ while the uniform attachment generator does not.

6 Conclusions and further prospects

In this paper, we have shown that, by proper scaling, the Gromov hyperbolic δ can be made relevant to finite diameter spaces. If the scaled δ of a finite diameter metric space is less than the maximum that it can achieve in Euclidean space, then the metric space is said to be *scaled Gromov-hyperbolic*. Because such spaces share the same fundamental metric property as negatively curved Riemannian manifolds, scaled Gromov hyperbolic spaces are expected to enjoy such archetypical properties as subquadratic isoperimetric inequality, bounded slimness, and the confinement of quasi-geodesics in an identifiable neighborhood of the geodesic. In [11], it is shown that negatively curved Busemann spaces, a closely related concept, enjoy these properties, and it is conjectured that these properties hold for the spaces introduced in this paper.

The 4-point condition is another way to define the Gromov-hyperbolic property and the same scaling of the 4-point δ leads to results in the same spirit as those developed here (see [12]). In particular,

$$\sup_{\Box\subset\mathbb{M}^3_{\kappa<0}} \frac{\delta_4(\Box)}{\text{diam}(\Box)} < \sup_{\Box\subset\mathbb{E}^3} \frac{\delta_4(\Box)}{\text{diam}(\Box)} = \frac{\sqrt{2}-1}{\sqrt{2}}$$

where $\Box$ denotes a quadruple of points (v^a, v^b, v^c, v^d). Contrary to Corollary 1, the left-most hand side of the inequality is neither left not right continuous at $\kappa = 0$.

Finally, the so-called algebra of bounded propagation operators has been shown to be relevant to infinite diameter Gromov hyperbolic graphs [23, Prop. 9.17]. In the same spirit as this paper, some indications as to how the bounded propagation concept could be made relevant to finite graphs are available in [14].

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Appendix: Proof of $c = \pm(a \pm b)$ in Proof of Theorem 2

Since the points x, y can be taken arbitrarily on $[v^b v^c], [v^a v^c]$, respectively, they can be chosen such that $d(x, v^c) = \lambda a$ and $d(y, v^c) = \lambda b$, for some $\lambda \in (0, 1)$. The hyperbolic law of cosines in $\triangle \tilde{x}\tilde{v^c}\tilde{y}, \triangle v^a v^b v^c \subset \mathbb{M}^2_{-1}$ reads, respectively,

$$\begin{aligned} \cosh \lambda c &= \cosh \lambda a \cosh \lambda b - \sinh \lambda a \sinh \lambda b \cos \tilde{\gamma} \\ \cosh c &= \cosh a \cosh b - \sinh a \sinh b \cos \bar{\gamma} \end{aligned}$$

By contradicting hypothesis, it is assumed that $\cos \tilde{\gamma} = \cos \bar{\gamma}$, so that

$$F_{a,b}(c) := f_{\lambda a, \lambda b}(\lambda c) - f_{a,b}(c) = 0$$

where

$$f_{a,b}(c) := \frac{\cosh a \cosh b - \cosh c}{\sinh a \sinh b}$$

It is easily seen that $F_{a,b}(\pm(a \pm b)) = 0$. Next, MAPLE symbolic software shows that $c = \pm(a \pm b)$ are the only real solutions [1], of which only $c = |a \pm b|$ are the relevant ones.

In case $\lambda = \frac{1}{2}$, a more direct way to get to the same four solutions is through the change of variable $p = 2 \cosh \lambda a$, $q = 2 \cosh \lambda b$, and $r = 2 \cosh \lambda c$, in which case the equation $F_{a,b}(c) = 0$ reduces to $r^2 - pqr + p^2 + q^2 - 4 = 0$. From the latter, it is an elementary exercise to show that there are only 4 real z-solutions, which traced back to the original parameters are $c = \pm(a \pm b)$.

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