49th IEEE Conference on Decision and Control
December 15-17, 2010
Hilton Atlanta Hotel, Atlanta, GA, USA

Geometry of Power Flow in Negatively Curved Power Grid

Reza Banirazi, *Member, IEEE*, and Edmond Jonckheere, *Fellow, IEEE*

***Abstract*—Motivated by the concept of the smart power grid, reliability of a power network is investigated from a topological viewpoint. Inspired from Riemannian geometry of manifolds, it is claimed that extreme load at specific parts of a large power grid can occur as a consequence of the local negative curvature in its hidden metric space. This paper contributes to four areas: (1) It draws a new course in the topological study of a power grid, which is, unlike most previous studies, in accordance with the electrical characteristic, not the topographical structure, of the power grid. (2) It extends the Riemannian geometry metaphor developed for data communication networks to the power grid, an area it has never pervaded. (3) It develops a unifying approach to deal with power and data networks, precisely at a juncture where the smart grid is bringing the two networks together. (4) It provides an analytical measure for the criticality of lines and stations in a bulk transmission system, which can find a place in reliability assessment and centralized flow control in the future smart grid.**

I. Introduction

Smart grid is a terminology with a disparate set of goals, which encompasses the entire electric power system, from the initial sources to the final consumers of energy. A broadly expanded transmission system, with large quantity of stochastic renewable power stations, is the conspicuity of the future smart grid. It means that the emerging transmission system should be resilient to larger disturbances and more distributed malfunctions, which means extreme *reliability* persistently remains the most important requirement for the transmission system. In a methodical view, the reliability of a complex system, like power grid, is defined as a function inversely proportional to the *criticality* of its subsystems, where the criticality of subsystems may be changed by modifying certain attributes. Then, the crucial problem becomes finding an appropriate measure for the criticality of lines and stations in a bulk transmission system.

In addition to the classical definition for the criticality of a power component as a function of its load and capacity, this paper claims that a deeper concept is the grid topology, say *grid curvature*, which highly affects the criticality of lines and stations. Under some conditions, the graph of a communication network, or that of the power grid for that matter, can be approximated by a Riemannian manifold. Then, the *graph curvature* can be defined as the curvature of its embedding manifold, and it is a fundamental Riemannian geometry paradigm that the geodesic flow is regulated by the curvature.

As a bridge between manifolds and graphs, the Gromov *Thin Triangle Condition* allows us to determine whether a graph is negatively curved in the very large scale. It has its inception in a property of a triangle drawn on a negatively curved surface to have the sum of its angles less than π, giving it a thin appearance. Moreover, the new concept of *scaled Gromov property* [1] provides us with an extension to some medium scale, and by the same token to the concept of nonnegatively curved graphs as well.

The aim of this paper is to extend the congestion analysis developed for negatively curved communication networks to a similar phenomenon in the power grid. We construct an innovative resistive network and infer the geometry of power flow in the transmission system from the topological structure of this resistive network. A crucial point here is that we investigate the topology of the power grid based on the *electrical* characteristic of power flows. The existing literature is mostly based on the *topographical* structure of the power grid, which, as correctly observed in [2], is not able to predict the electrical behavior of the network.

Furthermore, some recent attempts have been initiated to make a connection between power grid topology and cascade breakdowns leading to major blackouts, using some classical network concepts such as clustering and Small World [3], and degree distribution [4]. We claim that the Riemannian geometry approach to power grid topology is able to identify, in an *analytical* way, some architecture prone to create cascade blackouts from local faults. The domino effect in a negatively curved power grid can be analyzed in a Riemannian setup as follows: If a fault occurs in a line, this line is removed from the graph related to the power grid, with the consequence that the resulting graph is even more negatively curved, and so exacerbating the congestion problem with inevitable further faults. Accordingly, we offer a curvature-based analytical method to measure the criticality of lines and stations, which can be utilized in *reliability assessment* and *centralized flow control* in the future smart grid [5].

Due to space limitations, all the proofs have been omitted, but they are available through authors or in [6].

II. Riemannian Model of Traffic Congestion

A. Graph Theory

All graphs in this paper are assumed to be finite, undirected, connected, and simple, i.e., with no self-loops or multiple links. Assume graph $\mathcal{G}$ to be defined with a node (or vertex) set $\mathcal{V}$, and a link (or edge) set $\mathcal{E}$. Let us endow every link $x_i x_j \in \mathcal{E}$, between two incident nodes x_i and x_j, with a positive weight value w_{ij}, which represents the cost of traversal via that link. A *path* $p(x_k, x_l)$ between two arbitrary nodes x_k and x_l is a sequence of nodes, including x_k and x_l, with a link between any two successive ones and with no repeated node in the sequence. The *length* of a path is defined as the sum of the weights of the links traversed by that path. A path between x_k and x_l is a *geodesic* $[x_k x_l]$, if its length is minimum compared to all other possible paths joining x_k to x_l. Then, *geodesic distance* $d(x_k, x_l)$ is defined as the length of the geodesic path $[x_k x_l]$, i.e., $d(x_k, x_l) = \ell([x_k x_l])$. The distance d satisfies the triangle inequality in $\mathcal{G}$, that is,

This research was supported by NSF NetSE grant 1017881.

The authors are with Electrical Engineering Department, University of Southern California, CA 90089 (e-mail: {banirazi, jonckhee}@usc.edu).

$d(x_k, x_l) \le d(x_k, x_m) + d(x_m, x_l)$, and so $(\mathcal{G}, d)$ becomes a *metric space* in $\mathcal{G}$.

B. Curvature vs. Congestion

Traffic on a graph is driven by a simple traffic measure, which is the rate of commodities to be transmitted from a source s to a target t. Assume that the traffic routs on the graph in a least cost fashion, i.e., commodities are transferred from s to t along the geodesic $[st]$. Then, the main connections between congestion and curvature in a network can be formulated by the following definitions and theorems.

Definition 1 (Moment of Inertia and Graph Centroid): The moment of inertia of a connected weighted graph $\mathcal{G}$ with respect to a node x is defined a s $\phi(x) = \sum_{x_i} d^{\alpha}(x, x_i)$, for a constant $\alpha > 1$. Then, the centroid of $\mathcal{G}$ is defined as a node, for which the graph inertia becomes minimum.

Theorem 1 (Negative Curvature [7]): Consider a large, but finite, negatively curved graph, subject to uniformly distributed demand for commodities. Then, there are some specific nodes with very high traffic, which are the ones of least moment of inertia. If the graph is non-negatively curved, then both the traffic and inertia are more evenly distributed than in the case of a negatively curved graph. Furthermore, if the graph is positively curved with enough symmetry, both the traffic and inertia are uniformly distributed.

Graphs with absolute negative or positive curvature are, of course, the extreme situations. The real life networks are somewhere between these two extremes, with different local curvatures from negative to positive. In this case, a more detailed curvature analysis is required to determine the congestion points [8].

C. Riemannian Geometry of Traffic

The aim is to identify a manifold $\mathcal{M}^n$ and its Riemannian metric g, such that the dynamic nodes can be thought of as operating on the manifold. Precisely, the question is whether there exists an isometric map as $(\mathcal{G}, d) \to (\mathcal{M}^n, g)$. Ignoring all the mathematical complexity, we look at this problem very intuitively here. Let network $\mathcal{G}$ be a large ball $B_R(\mathbf{0})$ in some hyperbolic space $\mathbb{H}^n$, represented as the truncated Poincaré disk in Fig. 1. The routing between the source s and the target t is assumed to be in an optimal fashion for the hyperbolic metric. In this model, the *centroid* is the origin of the ball $B_R(\mathbf{0})$. To quantify the maximum congestion occurring at the centroid, consider the traffic load in a small ball $B_r(\mathbf{0})$, $r << R$, as $\Lambda_t(B_r(\mathbf{0})) = \int_{B_R(0)\times B_R(0)} \ell([s,t] \cap B_r(\mathbf{0}))\, ds\, dt$, where the integral is the total length of all traffic paths in the small ball, and as such it is a measure of the number of commodities in $B_r(\mathbf{0})$. It is argued that, no matter how theoretical our model of the traffic load $\Lambda_t(B_r(\mathbf{0}))$ is, it is remarkably accurate at confirming that the *load at the center* scales as N^2 where N is the number of nodes in a *real* network [7]. To reconcile the differential geometric and experimental approaches, it remains to show that, in an appropriately discretized version of $\Lambda_t(B_r(\mathbf{0}))$, for $B_r(\mathbf{0}) \subset B_R(\mathbf{0}) \subset \mathbb{H}^n$, the latter scales as N^2. In [7], the asymptotic formula in $\mathbb{H}^n$ is found as

$$\lambda_t(B_r(\mathbf{0})) := \frac{\Lambda_t(B_r(\mathbf{0}))}{\mathbf{vol}(B_R(\mathbf{0}))^2} = O(\text{constant}),$$

where 'constant' means 'independent of *R*.' Thus the traffic load $\Lambda_t(B_r(\mathbf{0}))$ in a small ball of measure $\mathbf{vol}(B_r(\mathbf{0}))$, near the centroid, scales as the square of the volume of the network $\mathbf{vol}(B_R(\mathbf{0}))^2$, which is modeled as the truncated hyperbolic manifold $B_R(\mathbf{0}) \subset \mathbb{H}^n$. In a tessellation of the Poincaré disk by polygons of equal areas, a node can be associated with each polygon, and $\mathbf{area}(B_R(\mathbf{0}))$ becomes the total number of nodes N. Similar arguments can be developed in n dimensions. Therefore, our model correctly *predicts* that the maximum traffic load scales as N^2. For a 2-dimensional Euclidean space, it is proved in [7] that

$$\lambda_t(B_r(\mathbf{0})) := \frac{\Lambda_t(B_r(\mathbf{0}))}{\mathbf{area}(B_R(\mathbf{0}))^2} = O(\frac{\text{constant}}{R}).$$

Therefore, up to some constant ($\doteq$), the traffic load at the center scales as

$$\frac{\mathbf{area}(B_R(\mathbf{0}))^2}{R} \doteq \mathbf{area}(B_R(\mathbf{0}))^{1.5} \doteq N^{1.5}.$$

More generally, in a *n*-dimensional Euclidean space, we have

$$\lambda_t(B_r(\mathbf{0})) := \frac{\Lambda_t(B_r(\mathbf{0}))}{\mathbf{vol}(B_R(\mathbf{0}))^2} = O(\frac{\text{constant}}{R^{n-1}}).$$

Hence, the traffic load at the center scales as

$$\frac{\mathbf{vol}(B_R(\mathbf{0}))^2}{\mathbf{vol}(B_R(\mathbf{0}))^{\frac{n-1}{n}}} \doteq \mathbf{vol}(B_R(\mathbf{0}))^{1+\frac{1}{n}} \doteq N^{1+\frac{1}{n}}.$$

With no doubt, as the dimension gets higher and higher, the Euclidean congestion decreases, and the gap between traffic loads in hyperbolic and Euclidean spaces increases.

Remark: While it is relatively easy to check the Gromov property of network, it is not easy at all to associate a dimension with a complex network. The remarkable feature is that the asymptotic traffic analysis *transcends the dimension*, at least in negatively curved spaces.

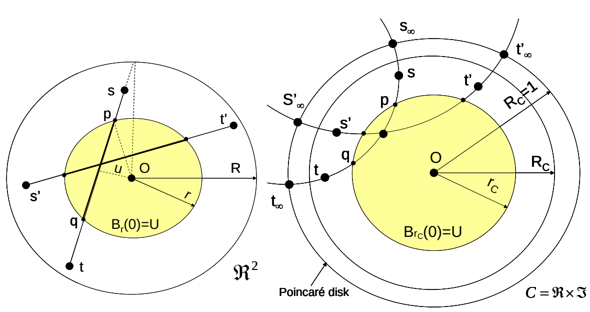

Fig. 1. "Traffic" on Euclidean (left) vs. Poincaré (right) space. The optimal paths are uniformly distributed in a Euclidean space, but maximally distributed at the center in a Poincaré space.

III. Resistive Graph vs. Transmission Grid

The first issue before applying the theorems and definitions of the previous section to the power grid is that the traffic should be expressed by such a simple variable as the rate of commodities passing through a node or a link. However, electrical power requires two variables to be identified, a generalized coordinate (charge) and a generalized force (voltage). Another challenge is that a commodity like a message has a specific header and is transferred from source to destination through an optimal path. However, electrical power, instead, flows along all transmission lines and stations from generating source to consuming loads in accordance with the power flow equations.

To overcome these problems and employ the powerful methods of differential geometry in the power flow problems, we introduce an *auxiliary resistive grid* in such a way that the geometry of power flow in the power grid can be concluded from the geometry of traffic in this auxiliary graph. Then, if the auxiliary graph is negatively curved, we expect to find some lines *vulnerable* to overloading, or some *critical* stations in the power grid. We intentionally separate stations from lines, since contrary to the routers in a data communication network, overloading hardly happens for the stations in a practical power grid; though, critical stations are still security weaknesses of the power grid.

A. Power Flow Equations

In the bus analysis of transmission system, Fig. 2, let us denote the voltages at buses k and m respectively with E_k and E_m relative to a ground reference voltage, and the line admittance between them with y_{km}. By Kirchhoff's law, the bus current I_k, defined as the net current drawn from bus k, is computed from $I_k = \sum_m y_{km}(E_k - E_m)$, where $y_{kk} = 0$. For a given voltage profile and network topology, the complex power transmitted from bus k to m is given by $S_{km} = P_{km} + jQ_{km} = E_k I_{km}^*$, where $I_{km}^* = y_{km}^*(E_k^* - E_m^*)$ is the complex conjugate of line current. With some mathematical manipulations, active and reactive power flows in the lines are obtained as

$$P_{km} = G_{km}V_k^2 - G_{km}V_kV_m \cos(\theta_k - \theta_m) + B_{km}V_kV_m \sin(\theta_k - \theta_m) \quad (1)$$

$$Q_{km} = B_{km}V_k^2 - B_{km}V_kV_m \cos(\theta_k - \theta_m) - G_{km}V_kV_m \sin(\theta_k - \theta_m) \quad (2)$$

where P_{km} and Q_{km} are respectively active and reactive power flow from bus k to m, V_k and θ_k voltage magnitude and phase angle at bus k with $E_k = V_k \exp(j\theta_k)$, and G_{km} and $-B_{km}$ line conductance and susceptance with $G_{km} - jB_{km} = y_{km}$. Writing the equations for all lines connected to bus k, the net complex power injection into the bus is computed from

$$S_k = \sum_m P_{km} + j\sum_m Q_{km} = E_k I_k^*.$$

B. Auxiliary Resistive Grid

A resistive graph $\mathcal{R}$ is defined by a set of resistors as its links, where the weight of each link is equal to its resistance, i.e., $w_{ij} = R_{ij} > 0$, for $i \neq j$. As a graph, the geodesic distance or *shortest path distance*, between two arbitrary nodes x_k and x_l in $\mathcal{R}$, is defined as the path with the least *apparent resistance* joining them, and denoted by $R(x_k, x_l)$. By the auxiliary resistive grid, we mean emulating the power grid with a resistive graph, such that the power flow in the power grid can be analogized with the electrical current in the resistive graph.

Based on physical properties of the power grid, operating in a normal steady-state mode, there is a weak coupling between (P,θ) and (Q,V) components. With this assumption, a pair of decoupled equations can approximate the fluctuation of line active and reactive power flows around their steady-state values as

$$P_{km} - \bar{P}_{km} = \overline{\left(\frac{\partial P_{km}}{\partial \theta_k}\right)} \cdot \left(\theta_k - \bar{\theta}_k\right) + \overline{\left(\frac{\partial P_{km}}{\partial \theta_m}\right)} \cdot \left(\theta_m - \bar{\theta}_m\right) \quad (3)$$

$$Q_{km} - \bar{Q}_{km} = \overline{\left(\frac{\partial Q_{km}}{\partial V_k}\right)} \cdot \left(V_k - \bar{V}_k\right) + \overline{\left(\frac{\partial Q_{km}}{\partial V_m}\right)} \cdot \left(V_m - \bar{V}_m\right) \quad (4)$$

where a bar on the top indicates steady-state value.

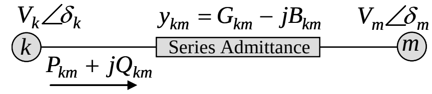

Fig. 2. Power grid transmission line model.

Another inherent characteristic of a practical power grid is that for most transmission lines, the line resistance is much smaller than the line reactance, $G_{km} << B_{km}$. Assuming $G_{km} \cong 0$, so-called lossless transmission system, (3) and (4) result in

$$\tilde{P}_{km} = \left(B_{km}\bar{V}_k\bar{V}_m \cos\bar{\theta}_{km}\right)\left(\tilde{\theta}_k - \tilde{\theta}_m\right) \quad (5)$$

$$\tilde{Q}_{km} = \left(2B_{km}\bar{V}_k - B_{km}\bar{V}_m \cos\bar{\theta}_{km}\right)\tilde{V}_k - \left(B_{km}\bar{V}_k \cos\bar{\theta}_{km}\right)\tilde{V}_m \quad (6)$$

where a tilde on the top indicates fluctuation around steady-state value, and $\bar{\theta}_{km} = \bar{\theta}_k - \bar{\theta}_m$.

To construct the auxiliary resistive grid, we also need to approximate $\bar{V}_k \cong \bar{V}_m \cos\bar{\theta}_{km}$. Indeed, this approximation is quite justifiable for the power grid operating in a steady-state mode, in which two incident buses are almost the same in voltage magnitude and phase angle, i.e. $\bar{V}_k \cong \bar{V}_m$ and $\cos\bar{\theta}_{km} \cong 1$. Then, (6) can be restated as

$$\tilde{Q}_{km} = \left(B_{km}\bar{V}_k\bar{V}_m \cos\bar{\theta}_{km}\right)\left(\frac{\tilde{V}_k}{\bar{V}_k} - \frac{\tilde{V}_m}{\bar{V}_m}\right). \quad (7)$$

Let $\tilde{S}_{km} = \tilde{P}_{km} + j\tilde{Q}_{km}$ represent the fluctuation of complex power transmitted from bus k to m. It follows from (5) and (7) that

$$\tilde{S}_{km} = \left(B_{km}\bar{V}_k\bar{V}_m \cos\bar{\theta}_{km}\right)\left[\left(\tilde{\theta}_k + j\tilde{V}_k/\bar{V}_k\right) - \left(\tilde{\theta}_m + j\tilde{V}_m/\bar{V}_m\right)\right] \quad (8)$$

Aggregating power fluctuations for all lines connected to bus k, the fluctuation of net complex power injection into bus k is obtained by $\tilde{S}_k = \sum_m \tilde{P}_{km} + j\sum_m \tilde{Q}_{km}$. Let us associate $\tilde{S}_{km}$ with a *complex current*, and $(\tilde{\theta}_k + j\tilde{V}_k/\bar{V}_k)$ with a *complex voltage*. Then, (8) can be interpreted as the voltage-current characteristic of a resistive graph with *variable* link conductance of $B_{km}\bar{V}_k\bar{V}_m \cos\bar{\theta}_{km}$, and nodal complex current sources of $\tilde{S}_k$.

Definition 2 (Auxiliary Resistive Grid): Consider a power grid in a steady-state power flow condition. The auxiliary resistive grid associated with this specific steady-state mode is a resistive graph *isomorphic* to the power grid, in which the resistance of each link is equal to $R_{km} = \left(B_{km}\bar{V}_k\bar{V}_m \cos\bar{\theta}_{km}\right)^{-1}$.

It is extremely important to note that all variables of the system, when evaluated in a specific steady-state mode, are assumed constant in short-time analysis, even though their values can change in medium- and long-time operation. Also notice that the root of fluctuations in the system is the variation of power supply/demand, happening in buses. Then, deviation of voltage magnitude $\tilde{V}_k$ and phase angle $\tilde{\theta}_k$ will be the consequence of those variations. Accordingly, the auxiliary resistive grid will only have current (complex power) sources in its nodes, where the value of each source is zero if there is no fluctuation in the power supply/demand of that node.

Theorem 2 (Resistive Graph vs. Transmission Grid): Consider the auxiliary resistive grid associated with a specific steady-state mode of the power grid. Let a set of complex current sources ψ_k injecting into the nodes, resulting in a set of node complex voltages U_k and a set of link complex currents J_{km}. Then, the fluctuation of line complex power and bus complex voltage in the power grid satisfy $\tilde{P}_{km} + j\tilde{Q}_{km} = J_{km}$ and $\tilde{\theta}_k + j\tilde{V}_k/\bar{V}_k = U_k$, iff the fluctuation of net complex power injection into the buses satisfy $\tilde{S}_k = \psi_k$.

C. Effective Resistance vs. Geodesic Distance

In a resistive graph, the current between two nodes x_k and x_l will not necessarily follow the path of least resistance $R(x_k, x_l)$. The relevant concept, instead, is that of effective resistance representing electrical distance. The effective resistance between two nodes is a measure of how electrically close they are, i.e., $R_{eff}(x_k, x_l)$ is small when there are many paths with low resistance between nodes x_k and x_l. The practical importance of the effective resistance, at least from the viewpoint of this study, is that the asymptotic behavior of the effective resistance $R_{eff}(x_k, x_l)$ versus the shortest path resistance $R(x_k, x_l)$ can provide information about the curvature of resistive graph.

Remark: The effective resistance satisfies the triangle inequality $R_{eff}(x_k, x_l) \le R_{eff}(x_k, x_m) + R_{eff}(x_m, x_l)$ in $\mathcal{R}$, i.e., R_{eff} is a *distance*, which is different from the geodesic distance R unless $\mathcal{R}$ is a tree. Hence, $(\mathcal{R}, R_{eff})$ becomes a *new* metric space different from $(\mathcal{R}, R)$.

Compute Effective Resistance: Given a resistive graph $\mathcal{R}(\mathcal{V}, \mathcal{E})$, the weighted *adjacency* matrix $\mathcal{A}$ is a symmetric square matrix of size $|\mathcal{V}|$, defined by $\mathcal{A}_{ij} = \mathcal{A}_{ji} = 1/R_{ij}$ if $x_i x_j \in \mathcal{E}$, and zero otherwise. Let $\mathcal{D}$ be the diagonal matrix with $\mathcal{D}_{ii} = \sum_{x_i x_j \in \mathcal{E}} 1/R_{ij}$. Then, the *weighted Laplacian* matrix is defined by $\mathcal{L} = \mathcal{D} - \mathcal{A}$. Let $\mathcal{L}_0$ be the sub-matrix obtained by deleting the row and column of $\mathcal{L}$ related to the reference node. It is easily seen that $\mathcal{L}_0$ is invertible and the effective resistance between two arbitrary nodes is given by $R_{eff}(x_k, x_l) = (\mathcal{L}_0^{-1})_{kk} + (\mathcal{L}_0^{-1})_{ll} - 2(\mathcal{L}_0^{-1})_{kl}$, where neither x_k nor x_l is the reference node, and by $R_{eff}(x_k, x_l) = (\mathcal{L}_0^{-1})_{kk}$, where x_l is the reference node.

IV. Geometry of Power Transmission

The aim of this section is to apply the abstract concepts outlined in Section II to the problem of power congestion in the power grid, using the resistive graph tools developed in Section III. The curvature concepts are being developed in order to anticipate which line, if any, is likely to overload. The intuitive concept is that the power grid will have congested spots if it is negatively curved. The *Gromov hyperbolic* property enters the scenery in the sense that, should the auxiliary grid be Gromov hyperbolic, the power grid is negatively curved and its electrical behavior is nearly isometric to a *core-centric* network.

A. Hyperbolic Resistive Grid

A possible node pattern is that of a Euclidean lattice, in which the nodes of the graph are obtained by the action of a discrete group of translations from a single node; and every two nodes, linked by a generator of the translation group, are connected by a link, i.e., a resistor in the resistive graph setting [9]. Familiar examples of Euclidean lattices include the square graph, the cubic graph, etc. One of the topological features of Euclidean resistive graphs, as stated by the following theorem, is that, remarkably, the asymptotic properties of the effective resistance differ from those of the shortest path resistance, and depend on the dimension of the lattice as well.

Theorem 3 (Euclidean Resistive Grid [10]): On a 1-dimensional Euclidean resistive string, $R_{eff}(x_k, x_l) = O(R(x_k, x_l))$; on a 2-dimensional Euclidean resistive lattice, $R_{eff}(x_k, x_l) = O(\log R(x_k, x_l))$; whereas on a 3-dimensional Euclidean resistive lattice, $R_{eff}(x_k, x_l) = O(1)$.

Now, we turn our attention to the opposite situation, where the effective resistance and the shortest path resistance have the same asymptotic behavior, independent of the dimension.

Given a resistive graph $\mathcal{R}$, a *geodesic triangle* $\Delta x_k x_l x_m$ is defined as a triangle made up of the shortest path resistances $R(x_k, x_l)$, $R(x_l, x_m)$, $R(x_m, x_k)$.

Definition 3 (Gromov Hyperbolic Graph): A resistive graph $\mathcal{R}(\mathcal{V}, \mathcal{E}, R)$ is Gromov hyperbolic if there exists a finite δ, such that every geodesic triangle $\Delta x_k x_l x_m$ has an inscribed triangle $\Delta x_{kl} x_{lm} x_{mk}$ of a perimeter not exceeding δ, that is, $R(x_{kl}, x_{lm}) + R(x_{lm}, x_{mk}) + R(x_{mk}, x_{kl}) \le \delta$.

Intuitively, a Gromov hyperbolic graph looks like a tree when *viewed at a distance*, where the concept of viewing at a distance is formalized in *large-scale geometry*, also referred to as *coarse geometry* [11]. The general idea of coarse geometry is that spaces which may locally be very different can still be very close on a large scale, and that many properties of them can be invariant under some coarse approximations, i.e., isometries up to some bounded distortion. As far as geometric techniques are concerned, we do not care about distortion, provided that it is uniformly bounded. In other words, in a resistive graph $\mathcal{R}$, the *exact* value of the effective resistance between nodes is irrelevant; but the relevant fact is whether this resistance is vanishing, finite, or infinite, which are coarse geometric invariants.

Definition 4 (Quasi-isometric Embedding): An embedding $f: \mathcal{G} \to \mathcal{H}$ of the graph $\mathcal{G} = (\mathcal{V}_G, \mathcal{E}_G, d_G)$ into the graph $\mathcal{H} = (\mathcal{V}_H, \mathcal{E}_H, d_H)$ is a *quasi-isometry*, if for every arbitrary nodes x_k and x_l in $\mathcal{G}$, there exist constants $\lambda \ge 1$, $\varepsilon \ge 0$, $c \ge 0$ such that

$$\frac{1}{\lambda} d_G(x_k, x_l) - \varepsilon \le d_H(f(x_k), f(x_l)) \le \lambda d_G(x_k, x_l) + \varepsilon$$

and every node in $\mathcal{H}$ has a distance at most c from some node in the image $f(\mathcal{G})$.

Remark: In the process of replacing a space with its quasi-isometric image, one of the most interesting features that remains invariant in large-scale is negative curvature, i.e., if one space is negatively curved, so is the other.

Theorem 4 (Quasi-isometric to Tree): Given a Gromov hyperbolic resistive graph $\mathcal{G} = (\mathcal{V}_G, \mathcal{E}_G, R_{effG})$ subject to a quasi-pole and a Cantor Gromov boundary, there exists a tree $\mathcal{T} = (\mathcal{V}_T, \mathcal{E}_T, R_{effT})$, finite constants $\alpha \ge 1$ and $\beta \ge 0$, and an embedding $f: \mathcal{G} \to \mathcal{T}$, such that $\forall x_k, x_l \in \mathcal{V}_G$,

$$\frac{1}{\alpha} R_{effG}(x_k, x_l) - \beta \le R_{effT}(f(x_k), f(x_l)) \le \alpha\, R_{effG}(x_k, x_l). \quad (9)$$

Theorem 4 justifies the fact that a Gromov hyperbolic graph, subject to some technical conditions, is isometric to a tree up to a bounded distortion. In other words, Gromov hyperbolic networks are a mathematical idealization of core-centric, negatively curved graphs.

Theorem 5 (Hyperbolic Resistive Grid): If the resistive graph $\mathcal{R} = (\mathcal{V}, \mathcal{E})$ is Gromov hyperbolic with a quasi-pole and a Gromov boundary, then for any $x_k, x_l \in \mathcal{V}$, we have $R_{eff}(x_k, x_l) = O(R(x_k, x_l))$. More precisely,

$$\lim_{R(x_k, x_l) \to \infty} \frac{R_{eff}(x_k, x_l)}{R(x_k, x_l)} \in [1/\alpha, \alpha],$$

where α is, as in (9), the multiplicative distortion in the quasi-isometry between the graph and its embedded tree [12].

B. Grid Curvature vs. Line Overload

To be practical, let us assign to each bus in the power grid an *operation risk factor* ρ, which represents the risk of experiencing power fluctuation in the bus. Then the *expecting value* of power fluctuation for each bus is determined in terms of the value of its power supply/demand in a specific steady-state operating mode. It means that if, for example, a bus operates under 1000 MVA power and risk factor 0.1, the transmission system must be reliable against 1000 × 0.1 = 100 MVA power increment/decrement in this bus.

To extend congestion analysis developed for communication networks to that of the power grid, we have hitherto dealt with two difficulties, i.e., defining traffic as a simple variable, and identifying the way in which this traffic is dispatched from source to destination. There are a couple of more challenges which are illustrated in Fig. 3 by comparison between congestion in the Internet and overload in the power grid. The first contrast is that in the Internet the limitation is on the routers with packet drops, whereas in the power grid the limitation is on the lines with overload trips. The second contrast is that in the Internet both send/receive and congestion occur in the nodes, whereas in the power grid supply/demand occurs in the nodes but overloading happens in the links.

To proceed, we need to introduce a distance between a line and a bus in the power grid. Although defining a notion of distance between these two dissimilar objects is mathematically controversial, we can do it in accordance with electrical intuition. Assume a current source with the value of I_l is injected into node x_l in the auxiliary resistive grid. This current increases the value of voltage in node x_k by $u_{km} = \left(\mathcal{L}_0^{-1}\right)_{kl} - \left(\mathcal{L}_0^{-1}\right)_{ml}$. Hence, link $x_k x_m$ receives a current with the value of $i_{km} = \left(\left(\mathcal{L}_0^{-1}\right)_{kl} - \left(\mathcal{L}_0^{-1}\right)_{ml}\right) \cdot \left(I_l / R_{km}\right)$, and $0 \le \left|i_{km} / I_l\right| \le 1$ represents a measure of *electrical closeness* between line $x_k x_m$ and bus x_l. Then, we define *weighted electrical centrality (inverse inertia)* for a link $x_k x_m$ by the sum of the weighted closeness between this link and all nodes in the auxiliary resistive grid, that is,

$$C_{km}(x_k x_m) = \frac{1}{(N-1)\cdot R_{km}} \sum_{x_i} \left(\left(\mathcal{L}_0^{-1}\right)_{ki} - \left(\mathcal{L}_0^{-1}\right)_{mi}\right) \cdot \left|S_i\right| \cdot \rho_i , \quad (10)$$

where $\left|S_i\right| = \sqrt{P_i^2 + Q_i^2}$ is the bus net apparent power, $0 \le \rho_i \le 1$ is the bus operation risk factor, and N denotes the number of nodes in the resistive grid.

Definition 5 (Line Moment of Inertia): Consider a power grid in a steady-state power flow condition, with the auxiliary resistive grid, and a set of bus operation risk factors, associated with this operating mode. Then, the *normalized* moment of inertia of the power grid with respect to a line is defined as

$$\phi_{km}(x_k x_m) = \left(1 - \frac{C_{km}(x_k x_m)}{\max\limits_{x_i x_j} C_{ij}(x_i x_j)}\right)^3 . \quad (11)$$

Remark: The moment of inertia of each line is defined in accordance with a specific stationary mode of operation. Therefore, it is constant only in short-time analysis, and can highly change in medium- and long-time operation.

While the local curvature of power grid can hardly be determined, if possible at all, Theorem 5 provides a practical tool to check the hyperbolic property of the power grid from its auxiliary resistive grid. Then, the following corollary predicts the geometry of power transmission in a hyperbolic power grid according to the moment of inertia, which can be analytically determined for any line.

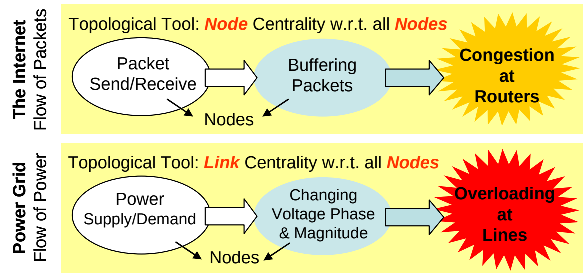

Fig. 3. Contrast between packet congestion in Internet and line overloading in power grid.

Corollary 1 (Negatively Curved Power Grid): Consider the auxiliary resistive grid associated with a specific steady-state mode of the power grid, with a set of bus operation risk factors. If this resistive grid is Gromov hyperbolic, then under uniform distribution of power fluctuations in buses, the lines with the lower moments of inertia experience more fluctuations in their transmitting power. On the other hand, if the auxiliary resistive grid is not hyperbolic, then the lines have nearly the same moment of inertia and the fluctuation of power tends to be distributed among them in nearly uniform manner.

C. Smart Power Scheduling/Routing

It is important to notice that a small moment of inertia implies higher vulnerability to uncertain changes in the power transmitted by a line, not necessarily overloading. To determine the line vulnerability to overloading, we need to know another characteristic of the line, so-called *line utilization*. In the simplest form, the line utilization is defined as $F_{km} = \left|S_{km}\right| / W_{km}$, where $\left|S_{km}\right| = \sqrt{P_{km}{}^2 + Q_{km}{}^2}$ is the line apparent power, and W_{km} denotes the rated capacity of the line in volt-amp (VA).

Corollary 2 (Reliable Transmission): Consider the negatively curved power grid in Corollary 1. Then for a reliable transmission under uniform distribution of disturbances in supply/demand, higher free capacity must be allocated to the lines with the lowest moments of inertia.

It is important to discriminate between the classical way of regarding high line utilization and the way proposed here. If a line is in high utilization, the red flag is already raised, even in a traditional dispatch. *However, the claim here is that for a line with respect to which the power grid has low moment of inertia, the red flag must be raised in quite a lower utilization compared to that in a traditional dispatch. Such a line, even with lower utilization, may be at higher risk of overloading in the presence of disturbance in supply/demand.*

Remark: In a negatively curved power grid, if the line utilization remains globally low for all possible scenarios of supply/demand, a low moment of inertia should not be a concern for line overload, i.e., reliability against supply/demand disturbance. Nevertheless, if the grid is quasi-isometric to a tree, the line failure could cut the service to many consumers, i.e., still unreliable against structural disturbance.

V. EVALUATION RESULTS

We evaluated the theoretical results of the previous sections through the IEEE 300 bus case. Fig. 4 shows, for five specific nodes $\boldsymbol{x}_k$ as samples, the ratio of the shortest path resistance $R(\boldsymbol{x}_k, \boldsymbol{x}_i)$ to the effective resistance $R_{eff}(\boldsymbol{x}_k, \boldsymbol{x}_i)$, each of which calculated between that node and all other nodes in the auxiliary resistive grid. This quantity, as discussed in Theorem 5, is a measure of hyperbolicity for the network. Two curves show a hyperbolic characteristic, in which R / R_{eff} is saturated despite the increase in R, which is symptomatic of a negatively curved resistive grid. The other curves, instead, reveal more like Euclidean resistive patterns, though different quantitative curvatures can be associated with them.

Fig. 5 compares the apparent power flow in different transmission lines together with the moment of inertia of each line. The operation risk factor is assumed the same for all buses. It clearly shows that there are some lines with low moment of inertia, operating under high power flow. However, as discussed in Corollary 2, without knowledge about the free capacities of different lines, we cannot give any measure about, or even compare their vulnerabilities, to overloading. Unfortunately, the rated capacities of lines are not given in the IEEE 300 bus case. Furthermore, assuming the same capacity for all lines is not practically justifiable and can be misleading. For example, one line under much lower operating power compared to another line may be more vulnerable to overloading, because of the combination of moment of inertia and free capacity of the lines.

Although, because of the lack of information, we cannot analyze the network reliability in a complete way, some conclusions can be drawn: (1) The grid is locally negatively curved with respect to some lines and buses. This observation is confirmed in Fig. 4 by the non-uniformity of the moment of inertia (Corollary 1), and in Fig. 5 by the symptoms of hyperbolicity in the auxiliary resistive grid (Theorem 5). (2) Number 1 line has zero moment of inertia with 458 MVA transmission load. This line is the only one connecting the power grid to its reference bus, namely *swing bus.* (3) Number 0 to 50 lines are in high centrality with respect to the fluctuations of power supply/demand in buses. To have a reliable power grid, these lines must operate quite far away their rated capacities. (4) Number 51 to 180 lines are in medium centrality, where a collection of lines with highest transmitting power occurs.

VI. CONCLUSION

This paper has targeted application of the congestion analysis developed for negatively curved communication networks to similar phenomena in power flow networks. As such, it is firstly a response to the *Robust Network Topology Dynamics Statement of Need* of the DoD [13], and secondly an initiative to state-of-the-art transmission systems for the future smart grids. We have investigated the geometry of the manifold underneath the network as a topological analysis tool to predict the behavior of the power grid in the presence of uncertain disturbances in load and generation. It is believed that this paper, beyond its application to the power grid, can open a new way to employ geometrical tools in revealing hidden behavior of power flow networks, in which the traffic cannot be quantified by only one variable.

REFERENCES

[1] E. Jonckheere, P. Lohsoont horn, and F. Bonahon, "Scaled Gromov Hyperbolic Graphs", *J. Graph Theory*, vol. 57, pp 157-180, 2008.

[2] P. Hines and S. Blumsack, "A Centrality Measure for Electrical Networks", *in The 41st. Hawaii International Conf. on System Sciences*, Hawaii, 2008.

[3] D. Pepyne, "Topology and Cascading Line Outages in Power Grids", *J. Systems Science and Systems Engineering*, 16(2):202-221, June 2007.

[4] P. Crucitti, V. Latora and M. Marchiori, "A Topological Analysis of the Italian Electric Power Grid", *Physica A*, vol. 338, pp. 92-97, 2004.

[5] *The Transmission Smart Grid Imperative, Developed for the U.S. Department of Energy, Office of Electricity Delivery and Energy Reliability*. by National Energy Technology Laboratory, September 2009.

[6] R. Banirazi and E. Jonckheere, "Geometry of Power Flow in Negatively Curved Power Grid: Toward a Smart Transmission System", *submitted to publish.*

[7] M. Lou. *Trafic Pattern Analysis in Negatively Curved Networks.* PhD thesis, University of Southern California, 2008.

[8] F. Ariaei, M. Lou, E. Jonckeere, B. Krishnamachari, and M. Zuniga, "Curvature of Sensor Network: Clustering Coefficient", *EURASIP J. Wireless Communications and Networlcing*, vol. 2008.

[9] H. Coxeter. *Introduction to Geometry.* Wiley Classics Library. John Wiley & Sons, New York, 1989.

[10] P. Barooah and J. Hespanha, "Graph Effective Resistances and Distributed Control: Spectral Properties and Applications", in *The 45th Conf. on Decision and Control*, December 2006.

[11] M. Gromov. *Metric Structures for Riemannian and Non-Riemannian Spaces*. vol. 152 of Progress in Mathematics. Springer-Verlag, 2001.

[12] E. Jonckheere, M. Lou, P. Barooah, and J. Hespanha, "Effective Resistance of Gromov Hyperbolic Graphs: Application to Asymptotic Sensor Network Problems", *in IEEE Conf. on Decision and Control*, pp 1453-1458, New Orleans, LA, December 2007.

[13] J. Hall, *Strategic Environmental Research and Development Program Statement of Need for FY08; Sustainable Infrastructure (SI) New Start.* by Department of Defense, SON Number SISON-09-4, November 2006.

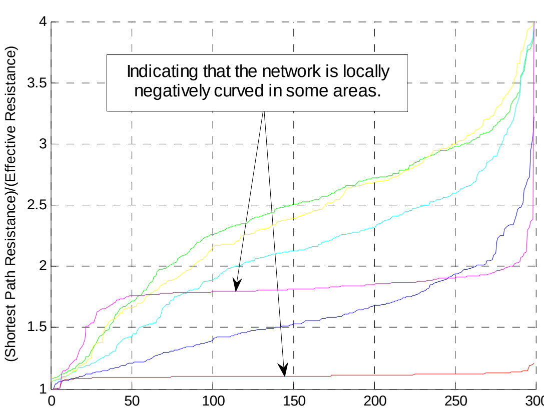

Fig. 4. $R(x, x_i)/R_{eff}(x, x_i)$ for five nodes in auxiliary resistive grid.

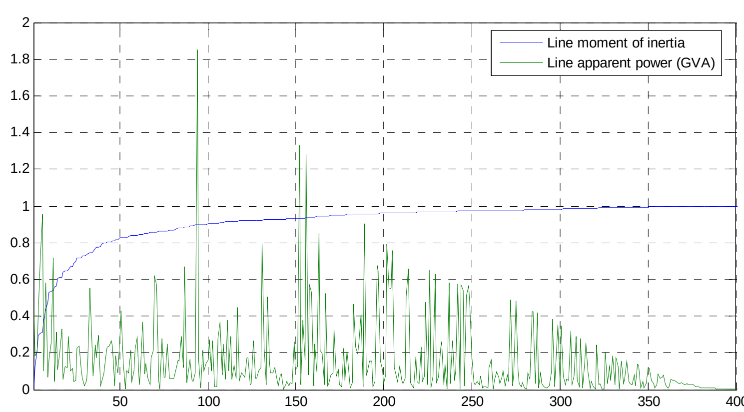

Fig. 5. Apparent power flow vs. moment of inertia for different lines in the power grid.