Upper bound on scaled Gromov-hyperbolic δ

E. A. Jonckheere*, P. Lohsoonthorn, and F. Ariaei
Dept. of Electrical Engineering–Systems
University of Southern California
Los Angeles, CA 90089-2563
jonckhee@usc.edu

Abstract

The Gromov-hyperbolic δ or "fatness" of a hyperbolic geodesic triangle, defined to be the infimum of the perimeters of all inscribed triangles, is given an explicit analytical expression in term of the angle data of the triangle. By a hyperbolic extension of Fermat's principle, the optimum inscribed triangle is easily constructed as the orthic triangle, that is, the triangle with its vertices at the feet of the altitudes of the original triangle. From the analytical expression of the optimum perimeter δ, a Tarski-Seidenberg computer algebra argument demonstrates that the δ, scaled by the diameter of the triangle, never exceeds 3/2 in a Riemannian manifold of constant nonpositive curvature. As probably the most important corollary, a finite metric geodesic space in which the ratio δ/diam is (strictly) bounded from above by 3/2 for all geodesic triangles exhibits the same metric properties as a negatively curved Riemannian manifold.

The specific applications targeted here are those involving such very large but finite graphs as the Internet and the Protein Interaction Network. It is indeed argued that negative curvature is the precise mathematical formulation of their visually intuitive core concentric property.

1 Introduction

Let $\mathcal{M}$ be a complete Riemannian manifold with distance d induced by the Riemannian metric. Let $A, B, C \in \mathcal{M}$; let $[AB]$, $[BC]$, $[CA]$ be shortest length geodesic arcs joining A to B, B to C, and C to A, respectively; their lengths are $d(A,B)$, $d(B,C)$, $d(C,A)$, respectively [14, Th. 1.4.8]; let $\triangle ABC$ denote the geodesic triangle made up of those arcs. The *fatness* of the geodesic triangle [19] is defined as the minimum perimeter of an arbitrary inscribed triangle:

$$\delta(\triangle ABC) := \inf \left\{ d(X,Y) + d(Y,Z) + d(Z,X) : \begin{array}{r} X \in [BC] \\ Y \in [AC] \\ Z \in [AB] \end{array} \right\}. \tag{1}$$

For a complete Riemannian manifold $\mathcal{M}$ of constant sectional curvature $\kappa < 0$, the geodesics $[AB], [BC], [CA]$ are guaranteed to be unique [13, p. 4], and the fatness of the geodesic triangles is bounded [19, pp. 84-85] in the sense that

$$\delta(\mathcal{M}) := \sup\{\delta(\triangle ABC) : A, B, C \in \mathcal{M}\} \le \frac{6}{\sqrt{-\kappa}} < \infty. \tag{2}$$

The intuition behind this result is that large scale geodesic triangles in hyperbolic space are "thin." The relevance of the so-called Gromov condition $\delta < \infty$ is that it provides a more primitive definition of negative curvature than the one traditionally formulated as the sum of the internal angles of an arbitrary geodesic triangle adding to less than π.

*Corresponding author, 3740 McClintock Ave, Room 306, Los Angeles, CA 90089-2563; jonckhee@usc.edu; FAX: (213) 821-1109.

To extend the definition of δ to a metric space $(\mathcal{X}, d)$, it is necessary that, given any two points $A, B \in \mathcal{X}$, there exists a geodesic $[AB]$ of length $d(A, B)$ joining them. In this case, $(\mathcal{X}, d)$ is said to be a *geodesic space* [3, p. 4]. By the Hopf-Rinow theorem, a complete Riemannian manifold is geodesic [14, Th. 1.4.8], [3, Corollary 3.20]. In a geodesic space $(\mathcal{X}, d)$, the fatness $\delta(\mathcal{X})$ can certainly be defined and we will say that $(\mathcal{X}, d)$ is δ-negatively curved iff $\delta(\mathcal{X}) < \infty$. If $\mathcal{G}$ is a locally finite graph in which every link (edge) is given a weight, the induced length distance d makes $(\mathcal{G}, d)$ a geodesic space, and under the condition $\delta(\mathcal{G}) < \infty$, the graph $\mathcal{G}$ is said to be *δ-negatively curved*.

The visual inspection of the Internet Service Provider (ISP) graph [11] and the Protein Interaction Network (PIN) of the yeast reveals their so-called "core-concentric" property. The latter can be mathematically reformulated as their large scale geodesic triangles being "thin," as the sides of the large triangles transit through the core ($\triangle XYZ$), hence making the Gromov analysis particularly attractive. One problem with the Gromov approach—a problem that this paper specifically addresses—is that the concept of δ-hyperbolic geodesic metric spaces hardly makes any sense for finite graphs, as every finite graph no matter how awesome its size has finite δ.

In a finite graph, a more relevant measure would be the δ of the triangles properly *scaled* by their diameters. This approach would be rigorously justified by the comparison geometry argument that, if there is an identifiable property of δ/diam in hyperbolic Riemannian space, this property, translated to graph language, would confer the graph hyperbolic properties. Toward that goal, the first purpose of this paper is to derive an explicit formula for $\delta(\triangle ABC)$, where $\triangle ABC$ is an arbitrary geodesic triangle in a hyperbolic Riemannian manifold specified by its internal angles α, β, γ. The symbolic MAPLE software is already used to manipulate hyperbolic trigonometry to the fatness formula, but even more computationally intensive is the Tarski-Seidenberg decision that $\delta(\triangle ABC)/\text{diam}(\triangle ABC) \leq 3/2$, for any geodesic triangle in a Riemannian manifold of constant nonpositive sectional curvature.

With the above result, we can remedy the ill-definedness of δ-hyperbolic finite graphs. If we require $\delta(\triangle ABC)/\text{diam}(\triangle ABC) < 3/2$, then the graph behaves like a negatively curved Riemannian manifold. This latter concept has proved relevant in scale-free graphs [12], in multipath routing as a countermeasure to packet sniffing in communication networks [11], and in worm propagation and defense [7]. (See [6, 2] for some independent related work.)

An outline of the paper follows. The geodesic triangle $\triangle ABC$ is first assumed to have acute angles only. We begin in Section 2 by reviewing the Euclidean case, in which the problem of finding the minimum perimeter inscribed triangle is known as the *Fagnano problem* [5]. By the Fermat principle, the edges of the minimum perimeter triangle, once given a constant speed, bounce on the original triangle with reflection angles equal to the corresponding incidence angles. This orbit is easily constructed as the *orthic triangle*, that is, the triangle that has its vertices at the feet of the altitudes of $\triangle ABC$ (see Figure 1). We further propose the related Fejér construction as the one that lends itself most easily to a hyperbolic extension. In Section 3, the first order conditions are expressed as the *hyperbolic Fermat principle*, saying that, under the first order optimality conditions, the inscribed triangle has its incidence angles on the edges of $\triangle ABC$ equal to the corresponding reflection angles. At this stage, the question arises as to whether in hyperbolic space there exists an inscribed triangle with its incidence angles equal to its reflection angles. The result still holds true in hyperbolic geometry; specifically, the orthic triangle still yields the periodic orbit. This can be proved, synthetically, by a hyperbolic extension of the construction of Fejér. However, this does not provide analytical expressions of the angles involved and for this reason we go through some hyperbolic trigonometry manipulations, which make the bulk of Section 4, to rederive the result, along with analytical expressions of the incidence/reflection angles (see Eq. 3). As a by-product, we show that the altitudes of a hyperbolic triangle intersect at a single point (see Corollary 1). Finally, the second order conditions are presented as a trivial corollary of the hyperbolic Fejér construction. In Section 5, we derive the explicit fatness formula (see Eq. 4). In Section 6, we show that $\delta/\text{diam} \leq 3/2$. The obtuse angle case is quickly dealt with at the end of the paper in Section 7.

Finally, in Section 8, we argue that the negative curvature definition proposed here provides a unification of several recent concepts that have emerged in network sciences.

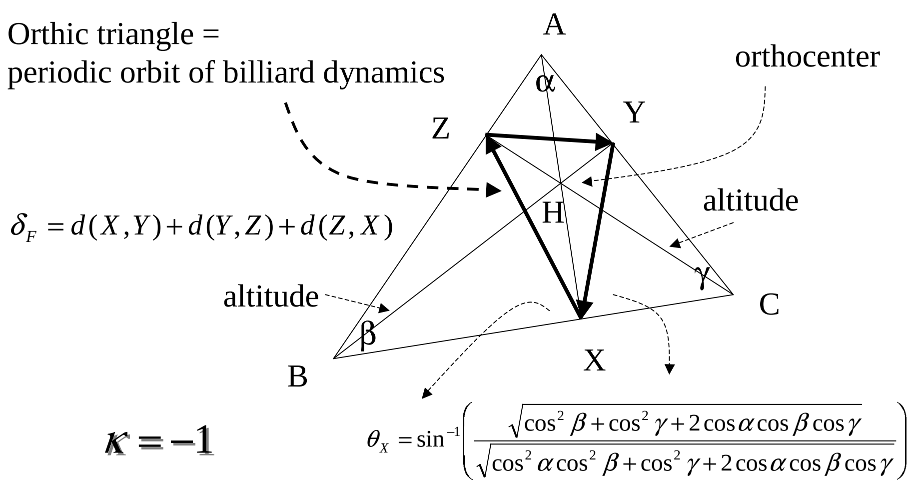

Figure 1: The Fagnano period orbit, shown to correspond to the orthic triangle. The same construction remains valid in constant curvature hyperbolic geometry.

2 Euclidean case

Let $\triangle ABC$ be a geodesic (rectilinear) triangle without obtuse angle in $\mathbb{E}^2$. Finding the minimum perimeter triangle inscribed to $\triangle ABC$ is the celebrated *Fagnano problem* [5], which has the following solution, depicted in Fig. 1: From A, draw the altitude $[AX]$, that is, the line segment such that $X \in [BC]$ and $[AX] \perp [BC]$. Likewise, draw the altitudes $[BY]$ and $[CZ]$. As is well known, the three altitudes intersect at a single point, referred to as *orthocenter* H. It turns out that $\triangle XYZ$, referred to as *orthic triangle* [5], is the minimum perimeter inscribed triangle.

The fact that the orthic triangle has minimum perimeter has various proofs, of various degrees of difficulty at extending them to the hyperbolic case. Among these proofs, we will retain the Fejér construction (see Figure 2): Fix $X \in [BC]$ and let us find $Z \in [BA]$, $Y \in [AC]$ such that $d(X,Y) + d(Y,Z) + d(Z,X)$ is minimized. Reflect X across $[AB]$ to get X'; likewise, reflect X across $[AC]$ to get X''. Clearly, $d(X,Y) + d(Y,Z) + d(Z,X) = d(X',Z) + d(Z,Y) + d(Y,X'')$, so that at optimality, X', Z, Y, X'' are aligned, which implies that $\angle BZX = \angle AZY$ and $\angle AYZ = \angle CYX$, that is, the Fermat Principle. To find the optimum X, observe that the angle at A of the isosceles triangle $X'AX''$ is twice $\angle BAC$, so that it does not depend on X. Clearly then $d(X', X'')$ is minimized iff $d(X', A) = d(A, X'') = d(A, X)$ is minimized, that is, $[AX] \perp [BC]$. A similar arguments holds true for the other points. Hence the orthic triangle yields the minimum perimeter inscribed triangle.

The fact that the minimum perimeter triangle has its reflection angles equal to the corresponding incidence angles, e.g., $\angle YXC = \angle ZXB$, is the traditional *Fermat principle* of geometrical optics [21]. Conversely, equality between the incidence and reflection angles is easily seen to be a first order condition for optimality. Besides optics, the billiard dynamics [16, Sec. 9.2] provides another metaphor, as the optimum inscribed triangle is the period 3 orbit of a ball bouncing on the edges of $\triangle ABC$.

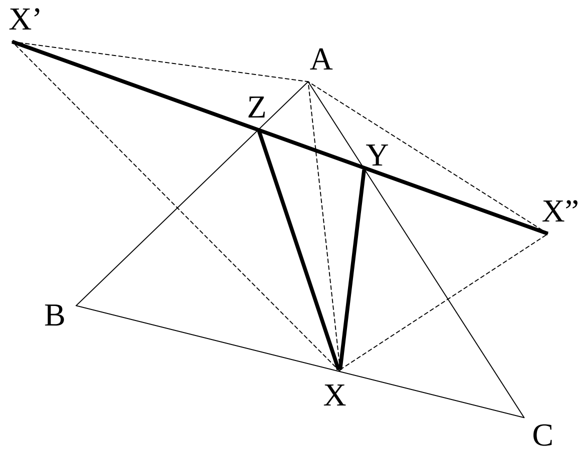

Figure 2: The Fejér construction, showing that the orthic triangle is the periodic three orbit. The construction remains valid in hyperbolic geometry.

3 Hyperbolic Fermat principle and first order conditions

Since we are working in a Riemannian manifold of constant negative sectional curvature, there is not much loss in generality in assuming that $\kappa = -1$ and so we will do to simplify the notation. Recall that such a $\kappa = -1$ space is said to be *hyperbolic* and is denoted as $\mathbb{H}$. Only the final and most important formulas are written for general negative curvature $\kappa < 0$.

Let $\triangle ABC$ be a geodesic triangle in the standard unit curvature hyperbolic space $\mathbb{H}$. This triangle is uniquely specified up to isometry by the three internal angles α, β, γ at the vertices A, B, C, respectively, provided that $\alpha+\beta+\gamma < \pi$. Let a, b, c be the lengths of the sides opposite to the angles α, β, γ, respectively. Let X, Y, Z be arbitrary points in $[BC], [CA], [AB]$, respectively.

Fix Y, Z and take X so as to minimize $d(Y, X) + d(X, Z)$. Under a first order perturbation of X, we can argue in the tangent space $T_X\mathbb{H}$, where Euclidean geometry prevails and hence the classical Fermat principle holds:

$$\cos(\angle YXC) = \cos(\angle ZXB) =: \cos(\theta_x).$$

We call this the *hyperbolic Fermat principle*, saying that a light ray emanating from Y, reflecting at $X \in [BC]$, to reach Z would have its reflection angle equal to its incidence angle. The same argument for the optimum Y, Z points yields

$$\begin{aligned} \cos(\angle ZYA) &= \cos(\angle XYC) =: \cos(\theta_y), \\ \cos(\angle XZB) &= \cos(\angle YZA) =: \cos(\theta_z). \end{aligned}$$

For the optimization problem to be a differentiable one, it is hence necessary that there exists an inscribed geodesic triangle $\triangle XYZ$ such that the reflection angles of its edges on the edges of $\triangle ABC$ equal the corresponding incidence angles. In the next section, we show that, if $\triangle ABC$ has acute angles only, such a triangle exists and is easily constructed as the orthic triangle.

4 Orthic triangle and second order conditions

It is easily seen that, for a hyperbolic geodesic triangle $\triangle ABC$, there exists a point $X \in [BC]$ such that $[AX] \perp [BC]$ if the angles $\angle ABX$ and $\angle ACX$ are acute. Therefore, if the triangle $\triangle ABC$ has no obtuse angle, there are points $X \in [BC]$, $Y \in [AC]$, $Z \in [AB]$ such that $[AX] \perp [BC]$, $[BY] \perp [AC]$, $[CZ] \perp [AB]$, respectively. Even though we do not know at present whether $[AX] \cap [BY] \cap [CZ] \neq \emptyset$, this construction yields an inscribed triangle $\triangle XYZ$, which, as we prove in the main body of this section, has the property that its reflection angles on the edges of $\triangle ABC$ equal the corresponding incidence angles.

We begin with some review of hyperbolic trigonometry:

Lemma 1 (Cosine Rule I) *In an arbitrary geodesic triangle $\triangle ABC$ in constant curvature $\kappa = -1$ Riemannian manifold, we have*

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma,$$

along with the formulas resulting from the permutations of the edges and the angles.

Lemma 2 (Cosine Rule II) *For an arbitrary geodesic triangle $\triangle ABC$ in hyperbolic space $\mathbb{H}$, we have*

$$\cosh (c) = \frac{\cos (\alpha) \cos (\beta) + \cos (\gamma)}{\sin (\alpha) \sin (\beta)},$$

along with the obvious permutations of the above.

Lemma 3 (Sine Rule) *For an arbitrary geodesic triangle $\triangle ABC$ in hyperbolic space $\mathbb{H}$, we have*

$$\frac{\sin \alpha}{\sinh a} = \frac{\sin \beta}{\sinh b} = \frac{\sin \gamma}{\sinh c}.$$

Lemma 4 (Pythagoras) *Given that $\triangle ABC$ is a geodesic triangle with internal angles $\alpha, \beta \leq \frac{\pi}{2}$ and $\gamma = \frac{\pi}{2}$ at the vertices A, B, C, respectively, in hyperbolic space $\mathbb{H}$, then the hyperbolic form of Pythagoras' theorem is given by the following formula:*

$$\cosh (c) = \cosh (a) \cosh (b).$$

In addition, the following relations hold:

$$\begin{aligned} \tanh (b) &= \sinh (a) \tan \beta, \\ \sinh (b) &= \sinh (c) \sin \beta, \\ \tanh (a) &= \tanh (c) \cos \beta. \end{aligned}$$

Let x, y, z be $d(B, X), d(C, Y), d(A, Z)$, respectively. Hyperbolic trigonometry in the right-angled subtriangles of $\triangle ABC$ yields

$$\begin{aligned} \tanh x &= \tanh c \cos \beta, \\ \tanh y &= \tanh a \cos \gamma, \\ \tanh z &= \tanh b \cos \alpha; \\ \tanh (a - x) &= \tanh b \cos \gamma, \\ \tanh (b - y) &= \tanh c \cos \alpha, \\ \tanh (c - z) &= \tanh a \cos \beta. \end{aligned}$$

From the Cosine Rule I applied to the triangles $\triangle ZBX$ and $\triangle YCX$ and Pythagoras' theorem, $d(Z, X)$ and $d(X, Y)$ can be expressed as follows:

$$\begin{aligned} \cosh d(Z, X) &= (\cosh (c - z) \cosh x - \sinh (c - z) \sinh x \cos \beta) \\ &= (\cosh (c - z) \cosh x) (1 - \tanh (c - z) \tanh x \cos \beta) \\ &= (\cosh (c - z) \cosh x) \left(1 - \tanh c \tanh a \cos^3 \beta\right), \end{aligned}$$

$$
\begin{aligned}
\cosh d(X,Y) &= (\cosh(a-x)\cosh y - \sinh(a-x)\sinh y\cos\gamma) \\
&= (\cosh(a-x)\cosh y)(1-\tanh(a-x)\tanh y\cos\gamma) \\
&= (\cosh(a-x)\cosh y)\left(1-\tanh a\tanh b\cos^3\gamma\right).
\end{aligned}
$$

Given that θ_x^l denotes $\angle ZXB$ and θ_x^r denotes $\angle YXC$, then the Sine Rule in the triangles $\triangle ZBX$ and $\triangle YCX$ yields the following results:

$$
\begin{aligned}
\sin^2\theta_x^l &= \left(\sin^2\beta\right)\frac{\sinh^2(c-z)}{\sinh^2 d(z,x)} = \left(\sin^2\beta\right)\frac{\sinh^2(c-z)}{\cosh^2 d(z,x)-1} \\
&= \left(\sin^2\beta\right)\frac{\sinh^2(c-z)}{(\cosh(c-z)\cosh x)^2(1-\tanh a\tanh c\cos^3\beta)^2-1}, \\
\sin^2\theta_x^r &= \left(\sin^2\gamma\right)\frac{\sinh^2(y)}{\sinh^2 d(x,y)} = \left(\sin^2\gamma\right)\frac{\sinh^2(y)}{\cosh^2 d(x,y)-1} \\
&= \left(\sin^2\gamma\right)\frac{\sinh^2(y)}{(\cosh(a-x)\cosh y)^2(1-\tanh a\tanh b\cos^3\gamma)^2-1}.
\end{aligned}
$$

Next, observe the following:

$$
\begin{aligned}
\sinh^2(c-z) &= \frac{\tanh^2(c-z)}{1-\tanh^2(c-z)} = \frac{(\tanh a\cos\beta)^2}{1-(\tanh a\cos\beta)^2}, \\
\cosh^2(c-z) &= \frac{1}{1-\tanh^2(c-z)} = \frac{1}{1-(\tanh a\cos\beta)^2};
\end{aligned}
$$

$$
\begin{aligned}
\sinh^2 y &= \frac{\tanh^2 y}{1-\tanh^2 y} = \frac{(\tanh a\cos\gamma)^2}{1-(\tanh a\cos\gamma)^2}, \\
\cosh^2 y &= \frac{1}{1-\tanh^2 y} = \frac{1}{1-(\tanh a\cos\gamma)^2};
\end{aligned}
$$

$$
\begin{aligned}
\cosh^2 x &= \frac{1}{1-\tanh^2 x} = \frac{1}{1-(\tanh c\cos\beta)^2}, \\
\cosh^2(a-x) &= \frac{1}{1-\tanh^2(a-x)} = \frac{1}{1-(\tanh b\cos\gamma)^2}.
\end{aligned}
$$

Now, using the above expressions yields the following:

$$
\begin{aligned}
\sin^2\theta_x^l &= \left(\sin^2\beta\right)\frac{\frac{(\tanh a\cos\beta)^2}{1-(\tanh a\cos\beta)^2}}{\left(\frac{1}{1-(\tanh a\cos\beta)^2}\frac{1}{1-(\tanh c\cos\beta)^2}\right)(1-\tanh a\tanh c\cos^3\beta)^2-1} \\
&= \frac{\left(\sin^2\beta\right)(\tanh a)^2\left(1-(\tanh c\cos\beta)^2\right)}{\left(\tanh^2 a-2\tanh a\tanh c\cos\beta+\tanh^2 c-\tanh^2 a\tanh^2 c\cos^2\beta+\tanh^2 a\tanh^2 c\cos^4\beta\right)}, \\
\sin^2\theta_x^r &= \left(\sin^2\gamma\right)\frac{\frac{(\tanh a\cos\gamma)^2}{1-(\tanh a\cos\gamma)^2}}{\left(\frac{1}{1-(\tanh b\cos\gamma)^2}\frac{1}{1-(\tanh a\cos\gamma)^2}\right)(1-\tanh a\tanh b\cos^3\gamma)^2-1} \\
&= \frac{\left(\sin^2\gamma\right)(\tanh a)^2\left(1-(\tanh b\cos\gamma)^2\right)}{\left(\tanh^2 a-2\tanh a\tanh b\cos\gamma+\tanh^2 b-\tanh^2 a\tanh^2 b\cos^2\gamma+\tanh^2 a\tanh^2 b\cos^4\gamma\right)}.
\end{aligned}
$$

Next, observe that, with the help of the Cosine Rule II, $\tanh a$, $\tanh b$, $\tanh c$ can be written as

$$\begin{aligned}\tanh^2 a &= \frac{\cosh^2 a - 1}{\cosh^2 a} \\ &= \frac{(\cos\beta\cos\gamma + \cos\alpha)^2 - \left(1-\cos^2\beta\right)\left(1-\cos^2\gamma\right)}{(\cos\beta\cos\gamma+\cos\alpha)^2} \\ &= \frac{\left(2\cos\alpha\cos\beta\cos\gamma + \cos^2\alpha + \cos^2\beta + \cos^2\gamma - 1\right)}{(\cos\beta\cos\gamma+\cos\alpha)^2},\end{aligned}$$

$$\tanh^2 b = \frac{\left(2\cos\alpha\cos\beta\cos\gamma + \cos^2\alpha + \cos^2\beta + \cos^2\gamma - 1\right)}{(\cos\gamma\cos\alpha+\cos\beta)^2},$$

$$\tanh^2 c = \frac{\left(2\cos\alpha\cos\beta\cos\gamma + \cos^2\alpha + \cos^2\beta + \cos^2\gamma - 1\right)}{(\cos\alpha\cos\beta+\cos\gamma)^2}.$$

Replacing $\tanh a$, $\tanh b$, and $\tanh c$ in the expressions for $\sin\theta_x^{l,r}$ by their values as given above yields

$$\sin^2\theta_x^l = \sin^2\theta_x^r = \frac{\cos^2\beta + \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma}{\cos^2\alpha + \cos^2\beta + \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma},$$

and finally,

$$\begin{aligned}\theta_x^l &= \theta_x^r \\ &= \arcsin\left(\frac{\sqrt{\cos^2\beta + \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma}}{\sqrt{\cos^2\alpha + \cos^2\beta + \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma}}\right).\end{aligned} \quad (3)$$

This proves that the reflection angle of the orthic triangle at X equals the incidence angle at the same point. The same fact is easily proved for the points Y, Z. Therefore, the orthic triangle is an inscribed triangle with its incidence angles equal the corresponding reflection angles. From the latter, the following emerges:

Corollary 1 *The altitudes of a constant curvature hyperbolic geodesic triangle with acute angles only intersect at a single point, called hyperbolic orthocenter.*

Proof. It follows from the preceding that the altitudes $[AX], [BY], [CZ]$ of the triangle $\triangle ABC$ are the angle bisectors of the triangle $\triangle XYZ$ and hence intersect in a single point. ■

Regarding the second order conditions, observe that the Fejér construction argument can be extended to hyperbolic geometry by substituting the hyperbolic geometry concept of inversion (which is conformal and hence preserves the angles [20, Th. 9]) for the Euclidean concept of reflection. From this synthetic geometry argument, it is clear that the orthocenter construction yields a (global) minimum. (See [17] for the explicit analytical argument involving the Hessian.)

5 Fatness formula

Given that $\triangle XYZ$ is the optimum inscribed triangle, the internal angles of $\triangle XYZ$ at X, Y, Z are $\pi - 2\theta_x, \pi - 2\theta_y, \pi - 2\theta_z$, respectively, where the θ's are derived from (3) as

$$\begin{aligned}\cos^2\theta_x &= \frac{\cos^2\alpha}{2\cos\alpha\cos\beta\cos\gamma + \cos^2\alpha + \cos^2\beta + \cos^2\gamma}, \\ \cos^2\theta_y &= \frac{\cos^2\beta}{2\cos\alpha\cos\beta\cos\gamma + \cos^2\alpha + \cos^2\beta + \cos^2\gamma}, \\ \cos^2\theta_z &= \frac{\cos^2\gamma}{2\cos\alpha\cos\beta\cos\gamma + \cos^2\alpha + \cos^2\beta + \cos^2\gamma}.\end{aligned}$$

Then

$$\begin{aligned}\cos(\pi-2\theta_x) &= 1-2\cos^2\theta_x \\ &= \frac{2\cos\alpha\cos\beta\cos\gamma+\cos^2\beta+\cos^2\gamma-\cos^2\alpha}{(2\cos\alpha\cos\beta\cos\gamma+\cos^2\alpha+\cos^2\beta+\cos^2\gamma)},\end{aligned}$$

$$\begin{aligned}\cos(\pi-2\theta_y) &= 1-2\cos^2\theta_y \\ &= \frac{2\cos\alpha\cos\beta\cos\gamma+\cos^2\alpha+\cos^2\gamma-\cos^2\beta}{(2\cos\alpha\cos\beta\cos\gamma+\cos^2\alpha+\cos^2\beta+\cos^2\gamma)},\end{aligned}$$

$$\begin{aligned}\cos(\pi-2\theta_z) &= 1-2\cos^2\theta_z \\ &= \frac{2\cos\alpha\cos\beta\cos\gamma+\cos^2\alpha+\cos^2\beta-\cos^2\gamma}{(2\cos\alpha\cos\beta\cos\gamma+\cos^2\alpha+\cos^2\beta+\cos^2\gamma)}.\end{aligned}$$

The Cosine Rule II for $\triangle XYZ$ yields the following results:

$$\begin{aligned}\cosh d(X,Y) &= \frac{\cos(\pi-2\theta_x)\cos(\pi-2\theta_y)+\cos(\pi-2\theta_z)}{\sin(\pi-2\theta_x)\sin(\pi-2\theta_y)} \\ &= \frac{\left(1-2\cos^2\theta_x\right)\left(1-2\cos^2\theta_y\right)+1-2\cos^2\theta_z}{(2\sin\theta_x\cos\theta_x)(2\sin\theta_y\cos\theta_y)}, \\ &= \frac{\left(2\cos^2\theta_x\cos^2\theta_y-\cos^2\theta_x-\cos^2\theta_y-\cos^2\theta_z+1\right)}{2(\sin\theta_x\cos\theta_x)(\sin\theta_y\cos\theta_y)}, \\ \sinh d(X,Y) &= \sqrt{\cosh^2 d(X,Y)-1} \\ &= \frac{\sqrt{(\cos^2\theta_x+\cos^2\theta_y+\cos^2\theta_z-1)^2-(2\cos\theta_x\cos\theta_y\cos\theta_z)^2}}{2(\sin\theta_x\cos\theta_x)(\sin\theta_y\cos\theta_y)},\end{aligned}$$

$$\begin{aligned}\cosh d(Y,Z) &= \frac{\left(2\cos^2\theta_y\cos^2\theta_z-\cos^2\theta_x-\cos^2\theta_y-\cos^2\theta_z+1\right)}{2(\sin\theta_y\cos\theta_y)(\sin\theta_z\cos\theta_z)}, \\ \sinh d(Y,Z) &= \frac{\sqrt{(\cos^2\theta_x+\cos^2\theta_y+\cos^2\theta_z-1)^2-(2\cos\theta_x\cos\theta_y\cos\theta_z)^2}}{2(\sin\theta_y\cos\theta_y)(\sin\theta_z\cos\theta_z)},\end{aligned}$$

$$\begin{aligned}\cosh d(Z,X) &= \frac{\left(2\cos^2\theta_z\cos^2\theta_x-\cos^2\theta_x-\cos^2\theta_y-\cos^2\theta_z+1\right)}{2(\sin\theta_z\cos\theta_z)(\sin\theta_x\cos\theta_x)}, \\ \sinh d(Z,X) &= \frac{\sqrt{(\cos^2\theta_x+\cos^2\theta_y+\cos^2\theta_z-1)^2-(2\cos\theta_x\cos\theta_y\cos\theta_z)^2}}{2(\sin\theta_z\cos\theta_z)(\sin\theta_x\cos\theta_x)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\sinh(d(X,Y)+d(Y,Z)+d(Z,X)) &= \cosh d(X,Y)\cosh d(Y,Z)\sinh d(Z,X) \\ &\quad +\cosh d(X,Y)\cosh d(Z,X)\sinh d(Y,Z) \\ &\quad +\cosh d(Y,Z)\cosh d(Z,X)\sinh d(X,Y) \\ &\quad +\sinh d(X,Y)\sinh d(Y,Z)\sinh d(Z,X) \\ &= \frac{\left(1-\cos^2\theta_x-\cos^2\theta_y-\cos^2\theta_z\right)}{2(\cos^2\theta_x)(\cos^2\theta_y)(\cos^2\theta_z)} \\ &\quad \cdot\sqrt{(\cos^2\theta_x+\cos^2\theta_y+\cos^2\theta_z-1)^2-(2\cos\theta_x\cos\theta_y\cos\theta_z)^2}.\end{aligned}$$

Finally, substituting $\cos\theta_x, \cos\theta_y, \cos\theta_z$ by their expressions in terms of α, β, γ yields

$$\begin{aligned}\sinh\left(d\left(X,Y\right)+d\left(Y,Z\right)+d\left(Z,X\right)\right) &= 2\sqrt{\left(2\cos\alpha\cos\beta\cos\gamma+\cos^2\alpha+\cos^2\beta+\cos^2\gamma\right)}\\ &\quad\cdot\sqrt{\left(2\cos\alpha\cos\beta\cos\gamma+\cos^2\alpha+\cos^2\beta+\cos^2\gamma-1\right)}\end{aligned}$$

and the expression for the fatness of a geodesic triangle in an arbitrary constant negative curvature space is as follows:

$$\delta\left(\triangle ABC\right)=\frac{1}{\sqrt{-\kappa}}\sinh^{-1}\left(\begin{array}{c}2\sqrt{\left(2\cos\alpha\cos\beta\cos\gamma+\cos^2\alpha+\cos^2\beta+\cos^2\gamma\right)}\\ \cdot\sqrt{\left(2\cos\alpha\cos\beta\cos\gamma+\cos^2\alpha+\cos^2\beta+\cos^2\gamma-1\right)}\end{array}\right). \tag{4}$$

Clearly, the maximum fatness triangle is obtained when $\alpha=\beta=\gamma=0$, that is, the ideal triangle, $\triangle$ideal, with its vertices at infinity. In the Poincaré unit disk model, this triangle has its vertices on the unit circle, 120 degrees apart. It follows that

$$\delta\left(\triangle\text{ideal}\right)=\frac{1}{\sqrt{-\kappa}}\sinh^{-1}\left(4\sqrt{5}\right)\approx\frac{1}{\sqrt{-\kappa}}2.887.$$

This indicates that the bound (2) is about twice as conservative as it should be.

6 Upper bound on scaled fatness

Recall that, by the Alexandrov Non Positively Curved (NPC) inequality [13, 2.3.1], the diameter of a triangle is achieved on an edge, so that it can be assumed, without loss of generality, that $\text{diam}(\triangle ABC)=a$. Then (4) along with the Cosine Rule I yields

$$\begin{aligned}\frac{\delta(\triangle ABC)}{\text{diam}(\triangle ABC)} &= \frac{\frac{1}{\sqrt{-\kappa}}\sinh^{-1}(2\sqrt{2\cos\alpha\cos\beta\cos\gamma+\cos^2\alpha+\cos^2\beta+\cos^2\gamma}}{\frac{1}{\sqrt{-\kappa}}\cosh^{-1}(\frac{\cos\beta\cos\gamma+\cos\alpha}{\sin\beta\sin\gamma})}\\ &\quad\cdot\sqrt{2\cos\alpha\cos\beta\cos\gamma+\cos^2\alpha+\cos^2\beta+\cos^2\gamma-1},\end{aligned} \tag{5}$$

where it is observed that the curvature terms drop. The next step is to derive a tight upper bound on the above expression, universal in negatively curved Riemannian manifolds.

6.1 Change of variables

We introduce the new variables

$$\begin{aligned}x &= 2\cos\alpha\cos\beta\cos\gamma+\cos^2\alpha+\cos^2\beta+\cos^2\gamma,\\ z &= \frac{\cos\beta\cos\gamma+\cos\alpha}{\sin\beta\sin\gamma},\end{aligned}$$

after which Equation (5) simplifies to

$$\frac{\delta(\triangle ABC)}{\text{diam}(\triangle ABC)}=\frac{\sinh^{-1}(2\sqrt{x(x-1)}}{\cosh^{-1}(z)}.$$

Converting the inverse hyperbolic trigonometric functions to natural logarithms,

$$\begin{aligned}\sinh^{-1}(2\sqrt{x(x-1)} &= \ln(2\sqrt{x(x-1)}+2x-1),\\ \cosh^{-1}(z) &= \ln(z+\sqrt{z^2-1}),\end{aligned}$$

and substituting the arguments of the logarithms for p and q, viz.,

$$\begin{aligned}p &= 2\sqrt{x(x-1)}+2x-1,\\ q &= z+\sqrt{z^2-1},\end{aligned} \tag{6}$$

it follows that Equation (5) simplifies to

$$\frac{\delta(\triangle ABC)}{\mathrm{diam}(\triangle ABC)} = \frac{\ln p}{\ln q},$$

so that $\frac{\delta(\triangle ABC)}{\mathrm{diam}(\triangle ABC)} \leq \frac{3}{2}$ reduces to

$$q^3 - p^2 \geq 0. \tag{7}$$

Substituting (6) in (7), we get

$$-8x^2 + 8x - 8x\sqrt{x(x-1)} + 4\sqrt{x(x-1)} + 4z^3 + 4z^2\sqrt{z^2-1} - 3z - \sqrt{z^2-1} - 1 \geq 0. \tag{8}$$

The above inequality holds iff

$$\frac{1}{2} - \frac{1}{4}\sqrt{8z^3 - 6z + 2} \leq x \leq \frac{1}{2} + \frac{1}{4}\sqrt{8z^3 - 6z + 2}. \tag{9}$$

6.2 Relationship between x and z

In the hyperbolic triangle $\triangle ABC$, the angles α, β and γ are acute and such that $\alpha + \beta + \gamma < \pi$. Taking the cosine of $\beta + \gamma < \pi - \alpha$ yields, after some manipulation, $z > 1$. Furthermore, from the definition of x and z, it is not hard to derive

$$x = \left(z^2 - 1\right)\sin^2\beta \sin^2\gamma + 1.$$

The above and $z > 1$ yield $1 < x < z^2$. In (9), we can see that the inequality $\frac{1}{2} - \frac{1}{4}\sqrt{8z^3 - 6z + 2} < 1$ is always true. Hence, Condition (9) reduces to

$$1 < x \leq \frac{1}{2} + \frac{1}{4}\sqrt{8z^3 - 6z + 2}. \tag{10}$$

Without loss of generality (and based on the definition of diameter), it can be assumed that $\alpha > \beta > \gamma$. Therefore,

$$0 < \sin^2\gamma < \frac{2z+1}{(z+1)^2} < \sin^2\alpha < 1.$$

Hence,

$$1 < x < \frac{2z^2 - z - 1}{(z+1)}\sin^2\beta + 1.$$

To check whether the inequality (8) holds, it is only necessary to verify the following:

$$\frac{2z^2 - z - 1}{(z+1)}\sin^2\beta + 1 \leq \frac{1}{2} + \frac{1}{4}\sqrt{8z^3 - 6z + 2}.$$

Simplifying the above inequality yields the following polynomial inequality in z:

$$8z^5 + C_4 z^4 + C_3 z^3 + C_2 z^2 + C_1 z + C_0 > 0, \tag{11}$$

where the coefficients are functions of $\cos\beta$:

$$\begin{aligned} C_0 &= -\cos^4\beta - 2\cos^2\beta + 1, & C_1 &= -2\cos^4\beta - 4\cos^2\beta - 4, \\ C_2 &= 3\cos^4\beta - 2\cos^2\beta - 15, & C_3 &= 4\cos^4\beta - 2, \\ C_4 &= -4\cos^4\beta + 8\cos^2\beta + 12. \end{aligned}$$

6.3 Sturm sequence

We have to examine whether the left-hand side of (11) has some sign change for $z \in (1, \infty)$. In full generality, this is a Tarski-Seidenberg decision problem. Here we approach this problem by computing the Sturm sequence, $\{P_i(z)\}$, of (11). The polynomials in the Sturm sequence, with their coefficients in $\mathbb{R}[\cos\beta]$, are evaluated at $z = 1 + \varepsilon$ ($\varepsilon \to 0$) and $z \to \infty$; this yields $\lim_{\epsilon\downarrow 0} P_i(1+\varepsilon) =: r(\cos\beta)$ and $P_i(\infty) =: s(\cos\beta)$. Computation of the various polynomials is done in a straightforward manner with the help of MAPLE$^{\mathrm{TM}}$; however, the resulting expressions are very long and relegated to [9].

6.4 results

The signs of $r(\cos\beta)$ and $s(\cos\beta)$ are determined as β changes in $[0, \frac{\pi}{2})$. The results are shown in the following table:

$P_i(z)$	sign of $r(\cos\beta)$ and $(s(\cos\beta))$					
$P_0(z)$	$+ \quad (+)$					
$P_1(z)$	$+ \quad (+)$					
$P_2(z)$	$+ \quad (+)$					
$P_3(z)$	β	$[0, .32)$	$[.32, .38)$	$[.38, .82)$	$[.82, 1.04)$	$[1.04, \frac{\pi}{2})$
		$+ \quad (+)$	$+ \quad (+)$	$+ \quad (+)$	$+ \quad (-)$	$- \quad (-)$
$P_4(z)$	β	$[0, .32)$	$[.32, .38)$	$[.38, .82)$	$[.82, 1.04)$	$[1.04, \frac{\pi}{2})$
		$+ \quad (+)$	$+ \quad (-)$	$- \quad (-)$	$- \quad (-)$	$- \quad (-)$
$P_5(z)$	$- \quad (-)$					

Since, for all β's, the number of sign changes both for $r(\cos\beta)$ and $s(\cos\beta)$ are equal, there are no real roots and hence no change of sign for (11). Since the sign of polynomial is positive as $z \downarrow 1$ and $z \to \infty$, it can be concluded that (11) is always true. This completes the proof of

Theorem 1 *For any geodesic triangle $\triangle ABC$ with acute angles only in a Riemannian space of constant curvature $\kappa < 0$,*

$$\frac{\delta(\triangle ABC)}{\mathrm{diam}(\triangle ABC)} \leq \frac{3}{2}.$$

In particular, the above property is an invariant of the curvature sign, not of the curvature.

It is also easily seen that, under the conditions of the above theorem, $\delta(\triangle ABC)/\mathrm{diam}(\triangle ABC) = 3/2$ can only be achieved for an infinitesimally small triangle. Therefore, the relevant negative curvature condition is $\delta(\triangle ABC)/\mathrm{diam}(\triangle ABC) < 3/2$, at a scale of triangles of a diameter bounded from below, viz., $\mathrm{diam}(\triangle ABC) > R > 0$.

The inequality $\delta(\triangle)/\mathrm{diam}(\triangle) < 3/2$ remains valid in an Alexandrov space [4, Chap. 4] of curvature uniformly bounded from above by $\kappa_{max} < 0$, but this requires a completely different argument [10].

As a corollary of Theorem 1, it is easy to see that $\delta(\triangle)/\mathrm{diam}(\triangle) > 3/2$ for triangles of nonvanishing diameters means positive curvature.

7 Obtuse angle case

Let $\triangle ABC$ be a triangle with $\alpha \geq \frac{\pi}{2}$ in hyperbolic space $\mathbb{H}$. This obtuse angle case is easily disposed of via a Fejér kind of argument. We treat both the Euclidean and hyperbolic cases simultaneously since they are the same. Let $\triangle XYZ$ be an inscribed triangle, fix $X \in [BC]$, and let us find $Z \in [AB]$, $Y \in [AC]$ such that the perimeter is minimized. Reflect X across $[AB]$, $[AC]$ to get X', X'', respectively, as shown in Fig. 3. Clearly,

$$d(X, Y) + d(Y, Z) + d(Z, X) = d(X', Z) + d(Z, Y) + d(Y, X''). \tag{12}$$

Next, it is claimed that

$$d(X', Z) + d(Z, Y) + d(Y, X'') \geq d(X', A) + d(A, X''). \tag{13}$$

Indeed, recall that in a nonpositive curvature space the distance function from the point X'' to an arbitrary point on the ray C/A is convex (see [13, Sec. 2.1]). Since A, Y are on the same side of the foot of the perpendicular from X'' to C/A, it follows that the distance function is monotone; hence $d(X'', A) \leq d(X'', Y)$. A similar argument yields $d(X', A) \leq d(X', Y)$. Hence $d(X', A) + d(X'', A) \leq d(X', Y) + d(X'', Y)$, from which a trivial appeal to the triangle inequality yields (13). Combining (12) and (13), it follows that optimality is reached for $Z = Y = A$. Hence $\min_{Z,Y} (d(X, Y) + d(Y, Z) + d(Z, X)) = 2d(A, X)$ and the latter is clearly minimum for $[AX] \perp [BC]$. Hence, $\delta(\triangle ABC)$ is twice the length of the altitude corresponding to

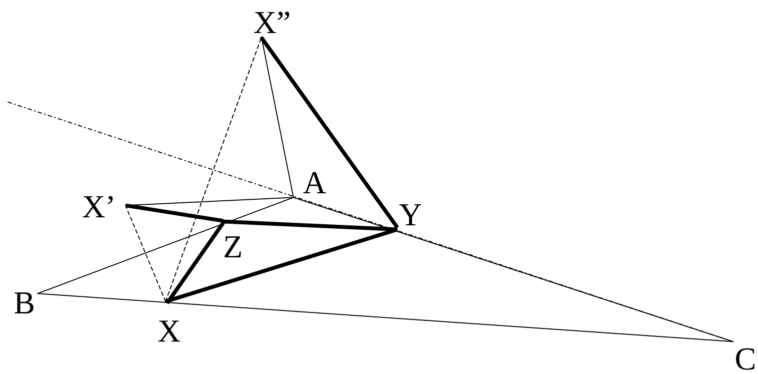

Figure 3: The Fejér construction for the obtuse angle case. The construction remains valid in hyperbolic geometry.

the obtuse angle. Quantitatively, from Pythagoras' theorem for $\triangle ABX$ and the Cosine Rule II for $\triangle ABC$, it follows that

$$\begin{aligned}\sinh\left(d\left(A,X\right)\right) &= \sin\beta\sinh c \\ &= \sin\beta\sqrt{\cosh^2 c-1} \\ &= \frac{1}{\sin\alpha}\sqrt{(2\cos\alpha\cos\beta\cos\gamma+\cos^2\alpha+\cos^2\beta+\cos^2\gamma-1)}.\end{aligned} \tag{14}$$

Therefore, the fatness formula for a geodesic triangle with obtuse angle α in constant negative curvature space $\kappa < 0$ is given by

$$\delta\left(\triangle ABC\right)=\frac{1}{\sqrt{-\kappa}}\sinh^{-1}\left(\frac{2}{\sin\alpha}\sqrt{(2\cos\alpha\cos\beta\cos\gamma+\cos^2\alpha+\cos^2\beta+\cos^2\gamma-1)}\right). \tag{15}$$

The scaled δ is given by

$$\frac{\delta(\triangle ABC)}{\mathrm{diam}(\triangle ABC)}=\frac{\sinh^{-1}\left(\frac{2}{\sin\alpha}\sqrt{(2\cos\alpha\cos\beta\cos\gamma+\cos^2\alpha+\cos^2\beta+\cos^2\gamma-1)}\right)}{\cosh^{-1}\left(\frac{\cos\beta\cos\gamma+\cos\alpha}{\sin\beta\sin\gamma}\right)}.$$

Theorem 2 *If $\triangle ABC$ is a geodesic triangle in constant nonpositive curvature space with $\alpha \geq \frac{\pi}{2}$, then*

$$\frac{\delta(\triangle ABC)}{\mathrm{diam}(\triangle ABC)} \leq 1.$$

Furthermore, the above is an invariant of the curvature sign, not of the curvature.

Proof. Consider the altitude $[AX]$ and extend it to the ray X/A. Pick a point A' "from X away from A" (see [5, p. 178]). As A' moves away but with $\angle BA'C \geq \frac{\pi}{2}$, $\mathrm{diam}(\triangle A'BC)$ remains constant while $\delta(\triangle A'BC)$

increases. Clearly, the maximum is reached when $\angle BA'C = \frac{\pi}{2}$. Setting $\alpha' = \alpha = \frac{\pi}{2}$ in the scaled δ formula yields

$$\frac{\delta(\triangle ABC)}{\mathrm{diam}(\triangle ABC)} = \frac{\sinh^{-1}\left(2\sqrt{(\cos^2\beta + \cos^2\gamma - 1)}\right)}{\cosh^{-1}\left(\frac{\cos\beta\cos\gamma}{\sin\beta\sin\gamma}\right)}.$$

Setting $x := \cos\beta$, $y := \cos\gamma$, the conjectured inequality $\delta(\triangle ABC) \leq \mathrm{diam}(\triangle ABC)$ becomes

$$4(x^2 + y^2 - 1)(1 - x^2)(1 - y^2) \leq (x^2 + y^2 - 1).$$

From $\gamma+\beta < \frac{\pi}{2}$, it is not hard to show that $x^2+y^2-1 > 0$, so that it remains to show that $4(1-x^2)(1-y^2) \leq 1$, equivalently, $2\sin\beta\sin\gamma \leq 1$. Under the inequality constraint $\beta + \gamma \leq \frac{\pi}{2}$, the former is easy to show by a Lagrange multiplier argument. ■

As a peripheral remark, observe that in the Poincaré disk model of $\mathbb{H}$, an obtuse-angled triangle need not have its altitudes intersecting in the disk[1].

8 Ubiquitous negative curvature property

Network science has so far been dominated by the, at point antagonistic [18], concepts of Small World and scale-free graphs. The Small World concept has promoted social networks to the status of hard science, while the (at point disputed) scale-free property of Internet graphs led to substantial development in heavy-tailed statistics.

Recent signs, however, point to a paradigm shift towards a new dichotomy: positive versus negative curvature. Simultaneously and independently, Jonckheere [11, 10, 12] and Baryshnikov [2] pointed out that scale-free Internet graphs are in fact negatively curved in the sense of the definition proposed in the present paper. At about the same time, Eckmann and Moses [6] proposed the clustering coefficient as the crucial parameter of the World Wide Web, which is an image of the social structure. They further suggest that the clustering coefficient provides a curvature measure, as low (high) clustering means negative (positive) curvature (see also [1]). Very recently, the need for an analysis, more global than the traditional ones, has manifested itself in the Protein Interaction Network [15]. As argued in [15], the scale-free concept is a local analysis in the sense that it deals with the degree, that is, the number of neighbors of a vertex, and it is unclear whether the heavy-tailed distribution of the degree is a truly global analysis. The clustering coefficient is already a step towards a more global analysis, as in this analysis not only the neighbors of a vertex, but also the way they are wired up, are taken into consideration [1]. The approach developed in this paper follows the same trend, as it is suggested that the condition $\delta(\triangle ABC)/\mathrm{diam}(\triangle ABC) < 3/2$ should be checked at the scale of large triangles. Along the same line of larger scale analysis of graphs of massive size, yet another recent development in biology proposes the "betweenness" as the crucial graph parameter [15]. The *betweenness* of a vertex is the (properly scaled) number of geodesic paths traversing that vertex. It is easily seen that high (low) betweenness means negative (positive) curvature. Indeed, the archtypical model of negatively curved graphs is the tree, for which $\delta(\triangle ABC)/\mathrm{diam}(\triangle ABC) = 0 < 3/2$; at the same time, a tree has high betweenness, as the betweenness of the root of a binary tree of depth n is $2^{2(n-1)}$. Conversely, low betweenness means positive curvature. Indeed, the archetypical model of a positively curved graph is the complete graph, for which $\delta(\triangle ABC)/\mathrm{diam}(\triangle ABC) = 2 > 3/2$; at the same time, the betweenness of any vertex of a complete graph vanishes.

Clearly, curvature is consistent with such recent trends as clustering and betweenness. However, because graph curvature mimics Riemannian geometry in which the curvature regulates the stability of the geodesics, the exponential growth of balls, etc., it is expected that curvature will become the all encompassing graph parameter.

9 Conclusion

While in coarse hyperbolic geometry, the only concern is whether $\delta(\triangle ABC) < \infty$ uniformly for all triangles, the hyperbolic geometry of very large but finite graphs requires a more precise estimate of $\delta(\triangle ABC)$,

[1]We thank Prof. M. Kapovitch, Univ. of Utah, for drawing our attention on this potential pitfall.

properly scaled relative to the diameter, along with a rule to determine how small the scaled δ should be for the finite graph to look like a Riemannian manifold of constant negative curvature. In this paper, it has been proved that, in the standard hyperbolic comparison space $\mathbb{H}$, $\delta(\triangle)/\text{diam}(\triangle) < 3/2$ for triangles of nonvanishing diameters, so that the latter provides the rule for arbitrary finite metric spaces. On a more general tone, we have the freedom to enforce the scaled δ-hyperbolic condition over a scale bounded from below, hence allowing some large scale analysis of finite graphs.

A slight variant of the proposed procedure consists in scaling relative to the perimeter, in which case the relevant condition becomes $\delta(\triangle ABC)/\text{perimeter}(\triangle ABC) < 1/2$ [10]. A somewhat more drastic departure consists in utilizing the properly scaled δ of the Gromov four-point condition [8], with results in the same spirit.

As early trends indicate, the scaled Gromov condition would be fundamental in coarse geometry of complex networks.

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